Finding the point of tangency between a curve and a line

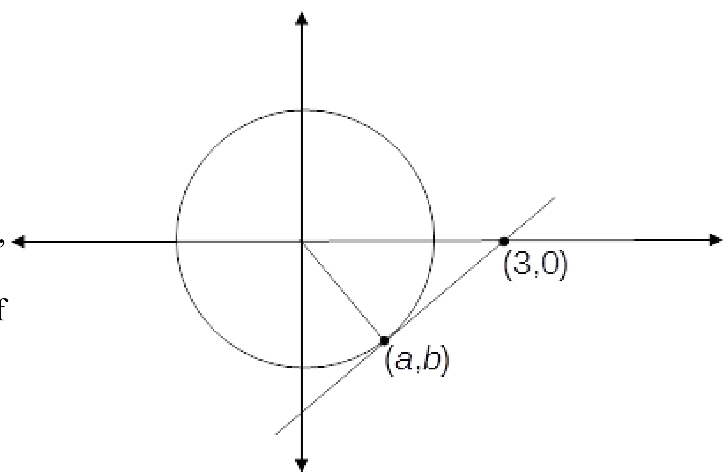
Problem:

Given that

a) a circle with radius 2 centered at the origin, and
b) a line which passes through the point $(3, 0)$

are tangent to each other in the third quadrant, find the point of tangency.

Solution:



Let (a, b) be the coordinates of the point of intersection.

Because (a, b) lies on the edge of the circle, it must satisfy $a^2 + b^2 = 4$. Solving for b, we have $b = \pm\sqrt{4 - a^2}$. Since the point of tangency is in the third quadrant, its coordinates are therefore $(a, -\sqrt{4 - a^2})$.

Next we calculate the distance between $(3, 0)$ and $(a, -\sqrt{4 - a^2})$:

$$d = \sqrt{(3-a)^2 - (0 + \sqrt{4 - a^2})^2} = \sqrt{9 - 6a + a^2 + 4 - a^2} = \sqrt{13 - 6a}$$

The tangent line, the radius of the circle, and the x-axis form a right triangle. Therefore, according to the Pythagorean thoerem:

$$\begin{gathered} d^2 + 2^2 = 3^2 \\ \left(\sqrt{13 - 6a}\right)^2 + 2^2 = 3^2 \\ 13 - 6a + 4 = 9 \\ 6a = 8 \\ a = \frac{4}{3} \end{gathered}$$

Substituting this into the expression for b, we get

$$b = -\sqrt{4 - a^2} = -\sqrt{4 - \left(\frac{4}{3}\right)^2} = -\sqrt{\frac{36}{9} - \frac{16}{9}} = -\sqrt{\frac{20}{9}} = -\frac{2\sqrt{5}}{3}$$

Thus, the point of intersection is

$$\boxed{\left(\frac{4}{3}, -\frac{2\sqrt{5}}{3}\right)}$$