7. Solve $\phi''(x) - c\phi(x) = 0$ on $0 < x < 1$
subject to the boundary conditions $\phi(0) = 0$ and $7\phi'(0) - 4\phi(1) = 0$

CASE 1: $c > 0$

General solution: $\phi(x) = A\sinh(x\sqrt{c}) + B\cosh(x\sqrt{c})$

BC1:

$B = 0$

$\longrightarrow \phi(x) = A\sinh(x\sqrt{c})$

BC2:

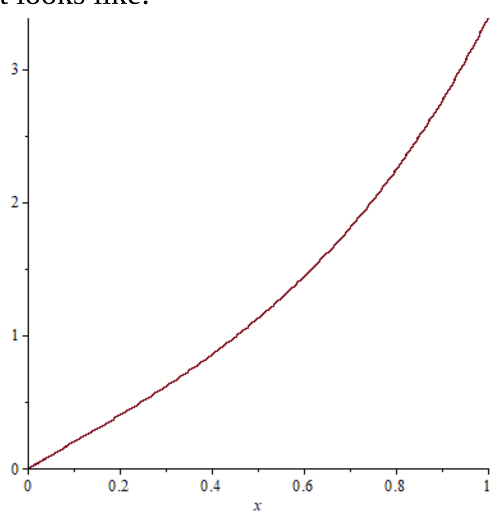
$7\sqrt{c} - 4\sinh(\sqrt{c}) = 0$

This equation must be solved numerically. It has one eigenvalue solution, $c \approx 3.7365$

There are no restrictions on A, so we have one eigenfunction

$\boxed{\phi(x) = A\sinh(x\sqrt{c})}$ with $\boxed{c \approx 3.7365}$

It looks like:



plot(subs({A=1, c=3.7365, phi),x=0..1)

CASE 2: c < 0

$\phi(x) = A\sin(x\sqrt{-c}) + B\cos(x\sqrt{-c})$

BC1:

$\phi(0) = 0$

$\longrightarrow B = 0$

$\longrightarrow \phi(x) = A\sin(x\sqrt{-c})$

BC2:

$7\phi'(0) - 4\phi(1) = 0$

$\longrightarrow A\sqrt{-c} - 4\sin(\sqrt{-c}) = 0$

This equation has no solutions for $c < 0$.

CASE 3: $c = 0$

$\phi(x) = Ax + B$

Since $\phi(0) = 0$, the y-intercept is zero and $\phi(x) = Ax$

Applying the second boundary condition, we have

$7\phi'(0) - 4\phi(1) = 0$

$\longrightarrow 7A - 4A = 0$

This is true only if A = 0, so the only solution is the trivial solution

$\boxed{\phi(x) = 0}$