5. Solve $\phi''(x) - c\phi(x) = 0$ on $0 < x < 1$

subject to the boundary conditions $\int_0^1 \phi(x) = 0$ and $\phi(0) = \phi(1)$

CASE 1: $c > 0$

General solution: $\phi(x) = A\sinh(x\sqrt{c}) + B\cosh(x\sqrt{c})$

BC1:

$$\int_0^1 \phi(x) = \left[\frac{A}{\sqrt{c}}\cosh(x\sqrt{c}) + \frac{B}{\sqrt{c}}\sinh(x\sqrt{c})\right]_0^1 = 0$$

$$\longrightarrow A\cosh(\sqrt{c}) + B\sinh(\sqrt{c}) - A = 0$$

BC2:

$$\phi(0) = \phi(1)$$

$$\longrightarrow B = A\sinh(\sqrt{c}) + B\cosh(\sqrt{c})$$

Combining the boundary conditions, we have:

$$\frac{A}{B} = \frac{-\sinh(\sqrt{c})}{\cosh(\sqrt{c}) - 1} = \frac{1 - \cosh(\sqrt{c})}{\sinh(\sqrt{c})}$$

$$\sinh^2(\sqrt{c}) = (\cosh(\sqrt{c}) - 1)^2$$

$$\cosh^2(\sqrt{c}) - 1 = (\cosh(\sqrt{c}) - 1)^2$$

Letting $p = \cosh(\sqrt{c})$,

$$p^2 - 1 = (p-1)^2$$

$$p^2 - 1 = p^2 - 2p + 1$$

$$p = 1 \longrightarrow \sqrt{c} = 0$$

Substituting into the general solution gives

$$\phi(x) = B$$

which satisifies the boundary conditions only when B = 0 (the trivial solution).

CASE 2: c < 0

$\phi(x) = A\sin(x\sqrt{-c}) + B\cos(x\sqrt{-c})$

BC1:

$$\int_0^1 \phi(x) = \left[-\frac{A}{\sqrt{-c}}\cos(x\sqrt{-c}) + \frac{B}{\sqrt{-c}}\sin(x\sqrt{-c})\right]_0^1 = 0$$

$\longrightarrow -A\cos(\sqrt{-c}) + B\sin(\sqrt{-c}) - (-A) = 0$

BC2:

$\phi(0) = \phi(1)$

$\longrightarrow B = A\sin(\sqrt{-c}) + B\cos(\sqrt{-c})$

Combining the boundary conditions,

$$\frac{A}{B} = \frac{\sin(\sqrt{-c})}{\cos(\sqrt{-c}) - 1} = \frac{1 - \cos(\sqrt{-c})}{\sin(\sqrt{-c})}$$

$-\sin^2(\sqrt{-c}) = (\cos(\sqrt{-c}) - 1)^2$

$\cos^2(\sqrt{-c}) - 1 = (\cos(\sqrt{c}) - 1)^2$

Letting $p = \cos(\sqrt{-c})$, we have

$p^2 - 1 = (p-1)^2$

$p^2 - 1 = p^2 - 2p + 1$

$p = 1 \longrightarrow \sqrt{-c} = 2\pi n$ for integer values of n

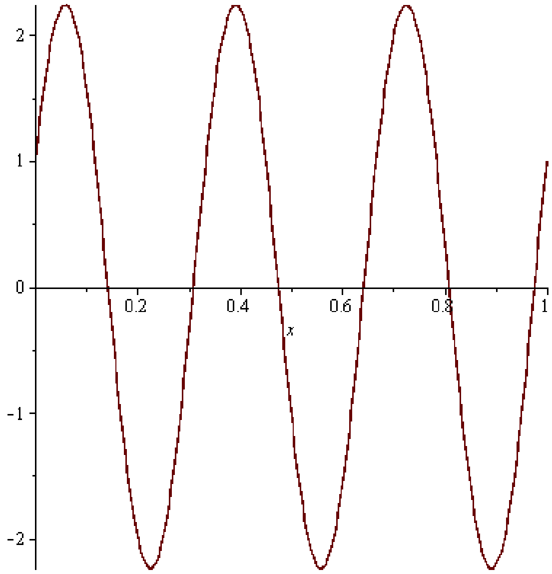
Substituting into the general solution gives

$$\boxed{\phi(x) = A\sin(2\pi n x) + B\cos(2\pi n x)}$$

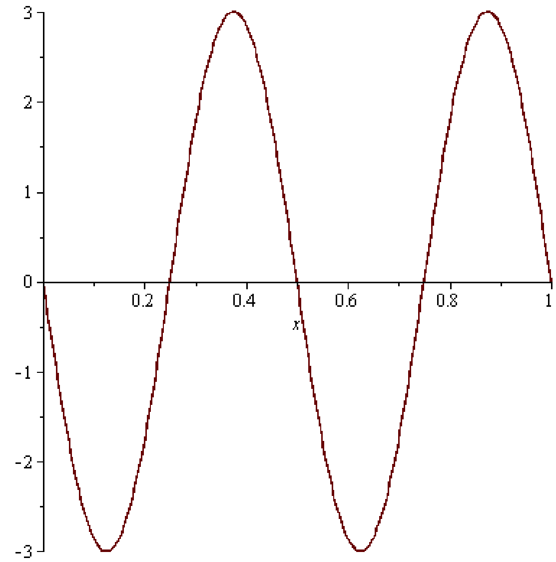
with corresponding eigenvalues $\boxed{c = -(2\pi n)^2}$ (Remember that $c < 0$.)

Since this function is periodic with period = 1 and since the integral of sine or cosine over one operiod is 0, this satisfies the boundary conditions for all values of A and B.

Here are some typical graphs:



plot(subs({A = 2, B = 1, n = 3}, phi), x = 0 .. 1)



plot(subs({A = -3, B = 0, n = 2}, phi), x = 0 .. 1)

CASE 3: $c = 0$

$\phi(x) = Ax + B$

This is a linear functions. Since $\phi(0) = \phi(1)$, the slope A must be zero.
Also, since the integral over $[0, 1] = 0$, B must also be zero. This gives only the trivial solution.