6. Solve $\phi''(x) - c\phi(x) = 0$ on $0 < x < 2$

subject to the boundary conditions $\displaystyle\int_0^2 \phi(x) = 0$ and $\phi(1) = 0$

CASE 1: $c > 0$

General solution: $\phi(x) = A\sinh(x\sqrt{c}) + B\cosh(x\sqrt{c})$

BC1:

$$\int_0^2 \phi(x) = \left[\frac{A}{\sqrt{c}}\cosh(x\sqrt{c}) + \frac{B}{\sqrt{c}}\sinh(x\sqrt{c})\right]_0^2 = 0$$

$$\longrightarrow A\cosh(\sqrt{2c}) + B\sinh(\sqrt{2c}) - A = 0$$

BC2:

$$\phi(1) = 0$$

$$\longrightarrow A\sinh(\sqrt{c}) + B\cosh(\sqrt{c}) = 0$$

Combining the boundary conditions, we have:

$$\frac{A}{B} = \frac{-\sinh(\sqrt{2c})}{\cosh(\sqrt{2c}) - 1} = -\frac{\cosh(\sqrt{c})}{\sinh\sqrt{c}}$$

Applying the double angle identities $\sinh(2x) = 2\sinh x\cosh x$ and $\cosh(2x) = 2\cosh^2 x - 1$,

$$= \frac{-2\sinh(\sqrt{c})\cosh(\sqrt{c})}{2\cosh^2(\sqrt{c}) - 2} = -\frac{\cosh(\sqrt{c})}{\sinh\sqrt{c}}$$

$$\longrightarrow \frac{\sinh(\sqrt{c})}{\cosh^2(\sqrt{c}) - 1} = \frac{1}{\sinh(\sqrt{c})}$$

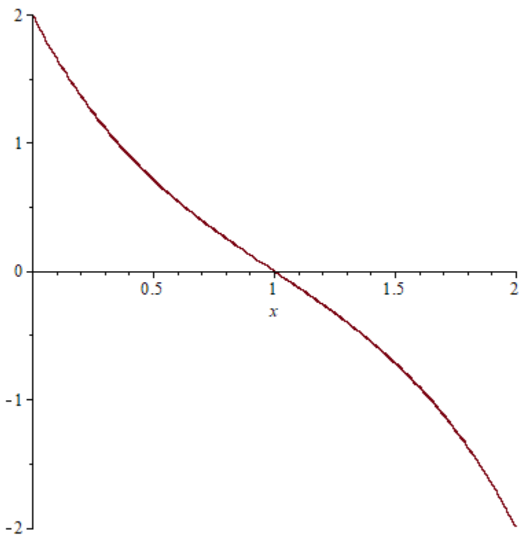
$$\longrightarrow \sinh^2(\sqrt{c}) = \cosh^2(\sqrt{c}) - 1$$

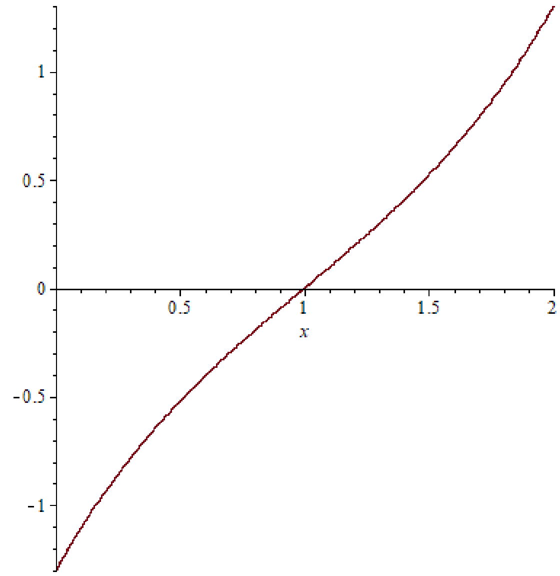
But this is just an identity (true for all values of $\sqrt{c}$), so it results in an infinite set of positive eigenvalues.

Substituting the boundary condition $\dfrac{A}{B} = -\dfrac{\cosh(\sqrt{c})}{\sinh(\sqrt{c})}$, derived earlier, into the general solution gives

$$\boxed{\phi(x) = -B\frac{\cosh(\sqrt{c})}{\sinh(\sqrt{c})}\sinh(x\sqrt{c}) + B\cosh(x\sqrt{c})} \quad \text{for all } \boxed{c > 0}.$$

Here are some typical graphs:





plot(subs({B = 2, c = 3}, phi), x=0..2)   plot(subs({B=-1.3,c=1.9, phi), x=0..2)

CASE 2: c < 0

$\phi(x) = A\sin(x\sqrt{-c}) + B\cos(x\sqrt{-c})$

BC1:

$$\int_0^2 \phi(x) = \left[-\frac{A}{\sqrt{-c}}\cos(x\sqrt{-c}) + \frac{B}{\sqrt{-c}}\sin(x\sqrt{-c})\right]_0^2 = 0$$

$\longrightarrow -A\cos(2\sqrt{-c}) + B\sin(2\sqrt{-c}) - (-A) = 0$

BC2:

$\phi(1) = 0$

$\longrightarrow A\sin(\sqrt{-c}) + B\cos(\sqrt{-c}) = 0$

Combining the boundary conditions,

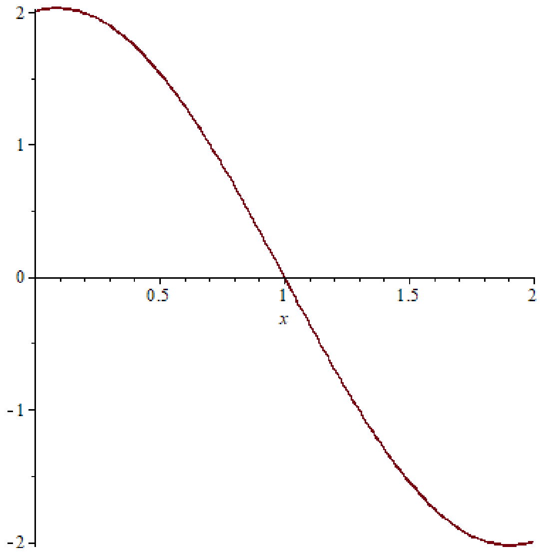
$$\frac{A}{B} = \frac{-\sin(2\sqrt{-c})}{1-\cos(2\sqrt{-c})} = -\frac{\cos(\sqrt{-c})}{\sin(\sqrt{-c})}$$

$$= \frac{2\sin(\sqrt{-c})\cos(\sqrt{-c})}{2\sin^2(\sqrt{-c})} = \frac{\cos(\sqrt{-c})}{\sin(\sqrt{-c})}$$

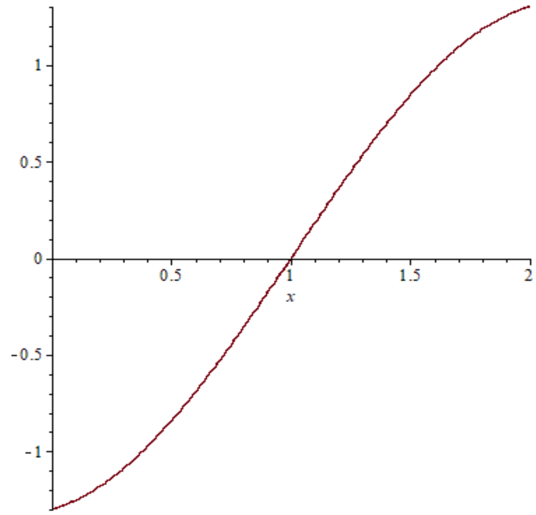
This is again an identity.

$$\boxed{\phi(x) = -B\frac{\cos(\sqrt{-c})}{\sin(\sqrt{-c})}\sin(x\sqrt{-c}) + B\cos(x\sqrt{-c})} \text{ for all } \boxed{c < 0}$$

Typical graphs are similar to those in the $c > 0$ case.



plot(subs(B = 2, c = -3, phi), x=0..2)



plot(subs({B = -1.3, c=-1.9, phi), x=0..2)

CASE 3: $c = 0$

$\phi(x) = Ax + B$

Since this function is linear has a *x*-intercept at $x = 1$, any line passing through the point $(1, 0)$ will satisfy the boundary conditions.

$\boxed{\phi(x) = A(x-1)}$, if $\boxed{c = 0}$