Problem #4 (25 points) - Another Regular Sturm-Liouville Problem

a.) (**15 points**) Determine the eigenvalues λ and corresponding non-zero eigenfunctions $\varphi(x)$ to the RSLP consisting of the ODE

$$\varphi''(x) + 4x\varphi'(x) + (4x^2 + 2 + \lambda)\varphi(x) = 0$$

for $0 < x < 1$ with the BCs $\varphi'(0) = 0$ and $\varphi'(1) = 0$. *Hint*: First show that

$$\varphi(x) = \begin{cases} Ae^{-x^2}\sinh(x\sqrt{-\lambda}) + Be^{-x^2}\cosh(x\sqrt{-\lambda}), & \text{when } \lambda < 0 \\ Axe^{-x^2} + Be^{-x^2}, & \text{when } \lambda = 0 \\ Ae^{-x^2}\sin(x\sqrt{\lambda}) + Be^{-x^2}\cos(x\sqrt{\lambda}), & \text{when } \lambda \neq 0 \end{cases}$$

for constants A and B, and be sure to check $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.

b.) (**5 Points**) Make plots of $\varphi_1(x)$, $\varphi_2(x)$ and $\varphi_3(x)$, showing that when the λ_n's are ordered as $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, the eigenfunction $\varphi_n(x)$ goes through zero exactly $n - 1$ times in the open interval $0 < x < 1$.

c.) (**5 Points**) If a piecewise continuous function $f(x)$ in the interval $0 < x < 1$ is expressed as

$$f(x) = \sum_{n=1}^{\infty} a_n\varphi_n(x)$$

for $0 < x < 1$, determine an expression for the a_n in terms of $f(x)$ and $\varphi_n(x)$.

a)

Since the solution has been provided, we just need to verify that that it satisifies the differential equation for each of the three cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. Differentiating as follows,

i) $\lambda < 0$

$$\varphi(x) = Ae^{-x^2}\sinh(x\sqrt{-\lambda}) + Be^{-x^2}\cosh(x\sqrt{-\lambda})$$

$$\varphi'(x) = -2Axe^{-x^2}\sinh(x\sqrt{-\lambda}) + Ae^{-x^2}\sqrt{-\lambda}\cosh(x\sqrt{-\lambda}) - 2Bxe^{-x^2}\cosh(x\sqrt{-\lambda}) + Be^{-x^2}\sqrt{-\lambda}\sinh(x\sqrt{-\lambda})$$

$$\varphi''(x) = -2Ae^{-x^2}\sinh(x\sqrt{-\lambda}) + 4Ax^2e^{-x^2}\sinh(x\sqrt{-\lambda}) - 4Axe^{-x^2}\sqrt{-\lambda}\cosh(x\sqrt{-\lambda}) - Ae^{-x^2}\lambda\sinh(x\sqrt{-\lambda})$$
$$-2Be^{-x^2}\cosh(x\sqrt{-\lambda}) + 4Bx^2e^{-x^2}\cosh(x\sqrt{-\lambda}) - 4Bxe^{-x^2}\sqrt{-\lambda}\sinh(x\sqrt{-\lambda}) - Be^{-x^2}\lambda\cosh(x\sqrt{-\lambda})$$

Substituting back into the original differential equation, we have

$$\underline{-2Ae^{-x^2}\sinh(x\sqrt{-\lambda})} + \underline{4Ax^2e^{-x^2}\sinh(x\sqrt{-\lambda})} - \underline{4Axe^{-x^2}\sqrt{-\lambda}\cosh(x\sqrt{-\lambda})} - \underline{Ae^{-x^2}\lambda\sinh(x\sqrt{-\lambda})}$$

$$\underline{-2Be^{-x^2}\cosh(x\sqrt{-\lambda})} + \underline{4Bx^2e^{-x^2}\cosh(x\sqrt{-\lambda})} - \underline{4Bxe^{-x^2}\sqrt{-\lambda}\sinh(x\sqrt{-\lambda})} - \underline{Be^{-x^2}\lambda\cosh(x\sqrt{-\lambda})}$$

$$\underline{-8Ax^2e^{-x^2}\sinh(x\sqrt{-\lambda})} + \underline{4Axe^{-x^2}\sqrt{-\lambda}\cosh(x\sqrt{-\lambda})} - \underline{8Bx^2e^{-x^2}\cosh(x\sqrt{-\lambda})} + \underline{4Bxe^{-x^2}\sqrt{-\lambda}\sinh(x\sqrt{-\lambda})}$$

$$+\underline{4Ax^2e^{-x^2}\sinh(x\sqrt{-\lambda})} + \underline{4Bx^2e^{-x^2}\cosh(x\sqrt{-\lambda})} + \underline{2Ae^{-x^2}\sinh(x\sqrt{-\lambda})} + \underline{2Be^{-x^2}\cosh(x\sqrt{-\lambda})}$$

$$+\underline{Ae^{-x^2}\lambda\sinh(x\sqrt{-\lambda})} + \underline{Be^{-x^2}\lambda\cosh(x\sqrt{-\lambda})} = 0$$

where the terms underlined in the same colors cance out. This confirms that the solution satisifies the DE.

ii) $\lambda = 0$

$$\varphi(x) = Axe^{-x^2} + Be^{-x^2}$$

$$\varphi'(x) = Ae^{-x^2} - 2Ax^2e^{-x^2} - 2Bxe^{-x^2}$$

$$\varphi''(x) = -6Axe^{-x^2} + 4Ax^3e^{-x^2} - 2Be^{-x^2} + 4Bx^2e^{-x^2}$$

Substituting into the original differential equation, we have

$$-6Axe^{-x^2} + 4Ax^3e^{-x^2} - 2Be^{-x^2} + 4Bx^2e^{-x^2} + 4Axe^{-x^2} - 8Ax^3e^{-x^2} - 4Bx^2e^{-x^2}$$
$$+4Ax^3e^{-x^2} + 4Bx^2e^{-x^2} + 2Axe^{-x^2} + 2Be^{-x^2} + Ax\lambda e^{-x^2} + B\lambda e^{-x^2} = 0$$

where the terms underlined in the same color cancel out and the terms containing λ disappear since $\lambda = 0$.

iii) $\lambda > 0$

$$\varphi(x) = Ae^{-x^2}\sin(x\sqrt{\lambda}) + Be^{-x^2}\cos(x\sqrt{\lambda})$$

$$\varphi'(x) = -2Axe^{-x^2}\sin(x\sqrt{\lambda}) + Ae^{-x^2}\sqrt{\lambda}\cos(x\sqrt{\lambda}) - 2Bxe^{-x^2}\cos(x\sqrt{\lambda}) - Be^{-x^2}\sqrt{\lambda}\sin(x\sqrt{\lambda})$$

$$\varphi''(x) = -2Ae^{-x^2}\sin(x\sqrt{\lambda}) + 4Ax^2e^{-x^2}\sin(x\sqrt{\lambda}) - 4Axe^{-x^2}\sqrt{\lambda}\cosh(x\sqrt{\lambda}) - Ae^{-x^2}\lambda\sin(x\sqrt{\lambda})$$
$$-2Be^{-x^2}\cos(x\sqrt{\lambda}) + 4Bx^2e^{-x^2}\cos(x\sqrt{\lambda}) + 4Bxe^{-x^2}\sqrt{\lambda}\sin(x\sqrt{\lambda}) - Be^{-x^2}\lambda\cos(x\sqrt{\lambda})$$

Substituting back into the original differential equation, we have

$$-2Ae^{-x^2}\sin(x\sqrt{\lambda}) + 4Ax^2e^{-x^2}\sin(x\sqrt{\lambda}) - 4Axe^{-x^2}\sqrt{\lambda}\cosh(x\sqrt{\lambda}) - Ae^{-x^2}\lambda\sin(x\sqrt{\lambda})$$
$$-2Be^{-x^2}\cos(x\sqrt{\lambda}) + 4Bx^2e^{-x^2}\cos(x\sqrt{\lambda}) + 4Bxe^{-x^2}\sqrt{\lambda}\sin(x\sqrt{\lambda}) - Be^{-x^2}\lambda\cos(x\sqrt{\lambda})$$
$$-8Ax^2e^{-x^2}\sin(x\sqrt{\lambda}) + 4Axe^{-x^2}\sqrt{\lambda}\cos(x\sqrt{\lambda}) - 8Bx^2e^{-x^2}\cos(x\sqrt{\lambda}) - 4Bxe^{-x^2}\sqrt{\lambda}\sin(x\sqrt{\lambda})$$
$$+4Ax^2e^{-x^2}\sin(x\sqrt{\lambda}) + 4Bx^2e^{-x^2}\cos(x\sqrt{\lambda}) + 2Ae^{-x^2}\sin(x\sqrt{\lambda}) + 2Be^{-x^2}\cos(x\sqrt{\lambda})$$
$$+Ae^{-x^2}\lambda\sin(x\sqrt{\lambda}) + Be^{-x^2}\lambda\cos(x\sqrt{\lambda}) = 0$$

where the terms underlined with the same color cancel out.

iv) Boundary conditions for the case where $\lambda < 0$:

The condition

$$\varphi'(0) = A\sqrt{-\lambda} = 0$$

implies that $A = 0$ so that $\varphi(x) = Be^{-x^2}\cosh(x\sqrt{-\lambda})$. The condition

$$\varphi'(1) = -2Be^{-1}\cosh(\sqrt{-\lambda}) + Be^{-1}\sqrt{-\lambda}\sinh(\sqrt{-\lambda}) = 0$$

implies that

$$\coth(\sqrt{-\lambda}) = \frac{\sqrt{-\lambda}}{2}$$

which has one solution when $\lambda_1 \approx -4.266$, shown here:

$$\varphi_1(x) = Be^{-x^2}\cosh(x\sqrt{-\lambda_1})$$

v) Boundary conditions for the case where $\lambda = 0$:

$$\varphi'(0) = Ae^0 - 2A(0)^2e^0 - 2B(0)e^0 = 0$$

implies $A = 0$

so that $\varphi(x) = Be^{-x^2}$

$$\varphi'(1) = -2B(1)e^{-1} = 0$$

implies that $B = 0$, so that this case gives only the trivial solution $\varphi(x) = 0$

vi) Boundary conditions for the case where $\lambda > 0$:

The condition

$$\varphi'(0) = A\sqrt{\lambda} = 0$$

implies $A = 0$

and that $\varphi(x) = Be^{-x^2}\cos(x\sqrt{\lambda})$.

The condition

$$\varphi'(1) = -2Be^{-1}\cos(\sqrt{\lambda}) - Be^{-1}\sqrt{\lambda}\sin(1\sqrt{\lambda}) = 0$$

implies that

$$\cot(\sqrt{\lambda}) = -\frac{\sqrt{\lambda}}{2}$$

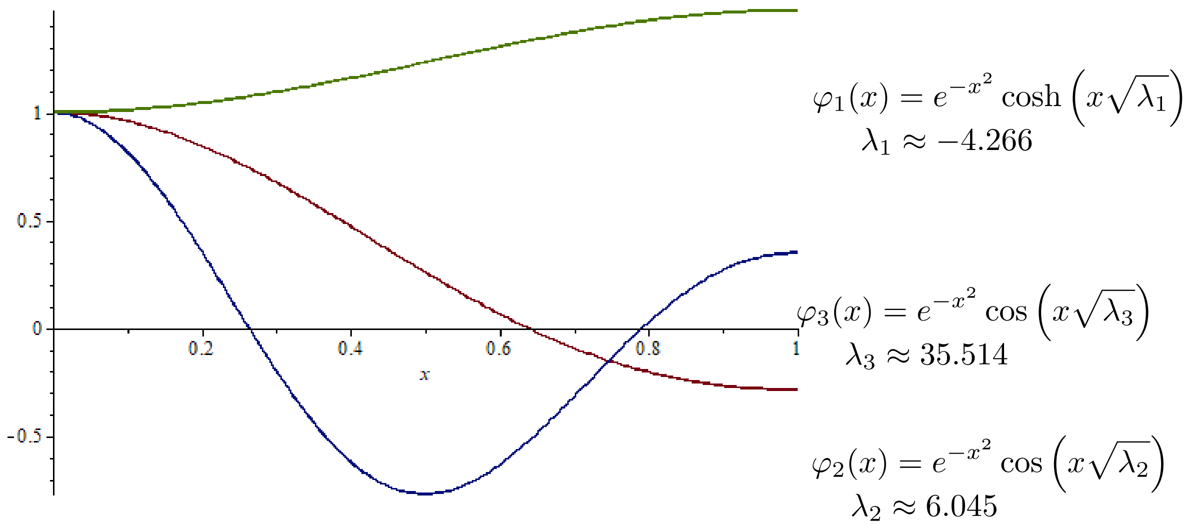
This has an infinite number of eigenvalue solutions

$\lambda_n \in \{6.045, 35.514, 84.842, \dots\}$

with corresponding eigenfunctions

$\varphi_n(x) = Be^{-x^2} \cos(x\sqrt{\lambda_n})$

b) Plots of eigenfunctions



We can see that when listed in order of increasing eigenvalues, the *n*th eigenfunction has $n-1$ zeros on the interval $(0, 1)$ as predicted by Sturm-Liouville theory.

c) Eigenfunction expansion of $f(x)$

In order to compute the coefficients a_n in

$$f(x) = \sum_{n=1}^{\infty} \varphi_n(x),$$

we need to take advantage of the orthogonality of the eigenfunctions with respect to the inner product

$$\langle \varphi_m, \varphi_n \rangle = \int_0^1 \varphi_m \varphi_n w(x) \mathrm{d}x$$

where the weight function $w(x)$ appears multiplied by $-\lambda\varphi(x)$ when the differential equation is written in standard Sturm-Liouville form.

$$(s(x)\varphi')' + p(x)\varphi = -\lambda w(x)\varphi$$

We can rewrite it in this form by multiplying both sides of the original differential equation by an integrating factor

$$\mu(x) = e^{\int 4x \mathrm{d}x} = e^{2x^2},$$

where the integrand ($4x$ in this case) is the function multplied by the $\varphi'(x)$ term.

This results in

$$e^{2x^2}\varphi''(x) + 4xe^{2x^2}\varphi'(x) + (4x^2+2+\lambda)e^{2x^2}\varphi(x) = 0\,,$$

or, after a bit of rearrangement

$$\left(e^{2x^2}\varphi'(x)\right)' + (4x^2+2+)e^{2x^2} = -\lambda e^{2x^2}\varphi(x)$$

So $w(x) = e^{2x^2}$.

Applying this “dot product” to $f(x)$ gives

$$\int_0^1 \varphi_n f(x)e^{2x^2}\,\mathrm{d}x = \sum_{n=1}^{\infty} a_n \int_0^1 \varphi_m\varphi_n e^{2x^2}\,\mathrm{d}x = a_n \int_0^1 \varphi_n^2 e^{2x^2}\,\mathrm{d}x$$

(since all the $m \neq n$ terms in the summation are equal to zero)

implying that

$$a_n = \frac{\int_0^1 \varphi_n f(x)e^{2x^2}\,\mathrm{d}x}{\int_0^1 \varphi_n^2 e^{2x^2}\,\mathrm{d}x}$$

where the integral in the denominator is given by

$$\int_0^1 \varphi_n^2 e^{2x^2}\,\mathrm{d}x = \int_0^1 B^2 e^{-2x^2}\cos^2(x\sqrt{\lambda_n})e^{2x^2}\,\mathrm{d}x = B^2\int_0^1 \cos^2(x\sqrt{\lambda_n})\mathrm{d}x$$

$$= \frac{B^2}{2}\int_0^1 \left(1+\cos(2x\sqrt{\lambda_n})\right)\mathrm{d}x = \frac{B^2}{2}\left(1+\frac{\sin(2\sqrt{\lambda_n})}{2\sqrt{\lambda_n}}\right)$$