Problem #4 (20 points) - A Regular Sturm-Liouville Problem

a.) **(10 points)** Determine the eigenvalues (λ_n) and eigenfunctions ($\varphi_n(x)$) for the differential equation

$$\varphi''(x) - 2x\varphi'(x) + (\lambda + x^2)\varphi(x) = 0$$

for $0 \le x \le 1$, along with the boundary conditions, $\varphi'(0) = \varphi(1) = 0$. *Hint*: See Problem #4 of Homework #4.

b.) **(10 points)** Determine the "dot" product for which $\varphi_m \cdot \varphi_n = 0$ when $m \neq n$ and use this to determine the coefficients a_n if the function

$$f(x) = e^{\frac{1}{2}x^2}$$

is expanded as

$$f(x) = e^{\frac{1}{2}x^2} = \sum_{n=1}^{\infty} a_n \varphi_n(x)$$

for $0 \le x \le 1$.

> *restart*

Here is the differential equation to be solved:

> $d := diff(\text{phi}(x), x\$2) - 2\cdot x\cdot diff(\text{phi}(x), x) + (x^2 + \text{lambda})\cdot\text{phi}(x) = 0$

$$d := \frac{d^2}{dx^2}\phi(x) - 2x\left(\frac{d}{dx}\phi(x)\right) + (x^2 + \lambda)\phi(x) = 0 \quad \textbf{(1)}$$

This can be rewritten in Sturm-Liouville form by use of an integrating factor e^{-x^2} :

> $d2 := diff(\exp(-x^2)\cdot diff(\text{phi}(x), x), x) + x^2\cdot\exp(-x^2)\cdot\text{phi}(x) = -\text{lambda}\cdot\exp(-x^2)\cdot\text{phi}(x)$

$$d2 := -2x\,e^{-x^2}\left(\frac{d}{dx}\phi(x)\right) + e^{-x^2}\left(\frac{d^2}{dx^2}\phi(x)\right) + x^2 e^{-x^2}\phi(x) = -\lambda e^{-x^2}\phi(x) \quad \textbf{(2)}$$

This is important because we need to know the weight function w(x) which appears on the right side of the equation as $-\lambda w(x)\,\phi(x)$. In this case, the weight function is the same as the integrating factor.

You can solve the differential equation by substituting a trial solution of the form

$y = e^{kx^2}$

into the original DE and solving for k. However, I am lazy, so I will let the computer do the work :

> $dsolve(d2, \text{phi}(x))$

$$\phi(x) = \_C1\, e^{\frac{1}{2}x\left(2\sqrt{-1-\lambda} + x\right)} + \_C2\, e^{\frac{1}{2}x\left(-2\sqrt{-1-\lambda} + x\right)} \quad \textbf{(3)}$$

>

This can be rewritten in a more visually appealing way by making use of Euler's formula. (Note that since we are interested in the interval (0, 1), the square root term is imaginary.

> $\text{phi} := C\cdot\exp\left(\frac{x^2}{2}\right)\cdot\cos(x\cdot\text{sqrt}(\text{lambda} + 1)) + D\cdot\exp\left(\frac{x^2}{2}\right)\cdot\sin(x\cdot\text{sqrt}(\text{lambda} + 1))$

$$\phi := C\,e^{\frac{1}{2}x^2}\cos\left(x\sqrt{1+\lambda}\right) + D\,e^{\frac{1}{2}x^2}\sin\left(x\sqrt{1+\lambda}\right) \quad \textbf{(4)}$$

Applying the initial conditions tells us that D = 0, and that $\sqrt{\lambda+1} = \pi\left(n-\frac{1}{2}\right)$ for $n = 1, 2, 3 \ldots$

Rewriting again with these substitutions gives us the final form of our eigenfunctions

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> phi := n→C[n]·exp(x^2/2)·cos(x·Pi·(n-1/2))
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$$\phi := n \rightarrow C_n\,e^{\frac{1}{2}x^2}\cos\left(x\,\pi\left(n-\frac{1}{2}\right)\right) \quad \textbf{(5)}$$

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>
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Here is a procedure to calculate the first 10 eigenfunctions

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> for n from 1 to 10 do:
   fun[n] := subs(C[n] = 1, phi(n))
  od
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$$fun_1 := e^{\frac{1}{2}x^2}\cos\left(\frac{1}{2}x\,\pi\right)$$

$$fun_2 := e^{\frac{1}{2}x^2}\cos\left(\frac{3}{2}x\,\pi\right)$$

$$fun_3 := e^{\frac{1}{2}x^2}\cos\left(\frac{5}{2}x\,\pi\right)$$

$$fun_4 := e^{\frac{1}{2}x^2}\cos\left(\frac{7}{2}x\,\pi\right)$$

$$fun_5 := e^{\frac{1}{2}x^2}\cos\left(\frac{9}{2}x\,\pi\right)$$

$$fun_6 := e^{\frac{1}{2}x^2}\cos\left(\frac{11}{2}x\,\pi\right)$$

$$fun_7 := e^{\frac{1}{2}x^2}\cos\left(\frac{13}{2}x\,\pi\right)$$

$$fun_8 := e^{\frac{1}{2}x^2}\cos\left(\frac{15}{2}x\,\pi\right)$$

$$fun_9 := e^{\frac{1}{2}x^2}\cos\left(\frac{17}{2}x\,\pi\right)$$

$$fun_{10} := e^{\frac{1}{2}x^2}\cos\left(\frac{19}{2}x\,\pi\right) \quad \textbf{(6)}$$

By plotting a few of them, we can see that they satisfy the initial conditions while having the requisite number of zeros for solutions to a Sturm-Liouville problem

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> plot({fun[1], fun[2], fun[3]}, x = 0..1)
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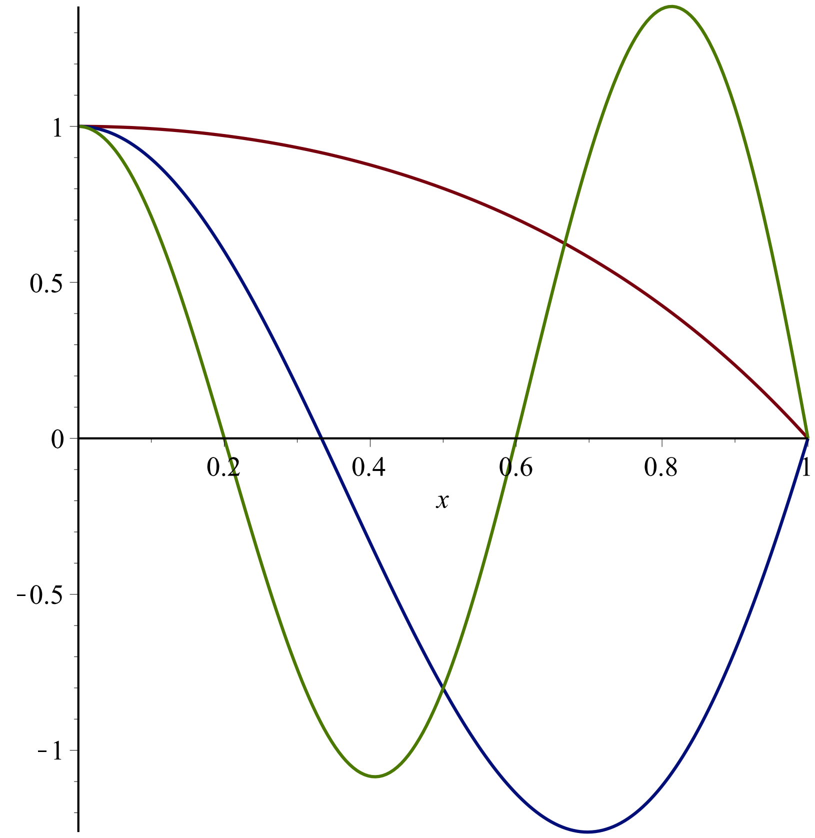

We can also verify the orthogonality condition holds. For example:

> $int(fun[3] \cdot fun[4] \cdot \exp(-x^2), x = 0..1)$

$$0 \quad (7)$$

> $int(fun[4] \cdot fun[4] \cdot \exp(-x^2), x = 0..1)$

$$\frac{1}{2} \quad (8)$$

We need to use a weighted sum of eigenfunctions to approximate f(x) = $e^{x^2/2}$

$$e^{x^2/2} = \sum_{n=1}^{\infty} C_n \varphi_n$$

Applying the inner product operator to both sides

$$\varphi_m \bullet f(x) = \int_0^1 \varphi_m f(x) e^{-x^2} dx,$$

our equation becomes

$$\int_0^1 \varphi_m e^{x^2/2}\, e^{-x^2}\, dx = C_n \int_0^1 \varphi_m\, \varphi_n\, e^{-x^2}\, dx.$$

Applying the orthogonality condition, which states

$$\int_0^1 \varphi_m\, \varphi_n\, e^{-x^2}\, dx = \begin{cases} 0 & m \neq n \\ x & x \geq a \end{cases},$$

and solving for $C_m = C_n$, we get eigenvalues of the form

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> for m from 1 to 30 do:
    C[m] := int(exp(x^2/2)·fun[m]·exp(-x^2), x=0..1) / int((fun[m])^2·exp(-x^2), x=0..1)
  od:
```

Actually, this integral is easy to do by hand since the exponential terms all cancel out. The solution can be written as:

```
> for m from 1 to 10 do:
    C[m] := 4/(Pi·(2·m-1))·(-1)^(m+1)
  od
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$$C_1 := \frac{4}{\pi}$$

$$C_2 := -\frac{4}{3\pi}$$

$$C_3 := \frac{4}{5\pi}$$

$$C_4 := -\frac{4}{7\pi}$$

$$C_5 := \frac{4}{9\pi}$$

$$C_6 := -\frac{4}{11\pi}$$

$$C_7 := \frac{4}{13\pi}$$

$$C_8 := -\frac{4}{15\pi}$$

$$C_9 := \frac{4}{17\pi}$$

$$C_{10} := -\frac{4}{19\pi} \qquad \textbf{(9)}$$

(which results in the same values as the integral equation)

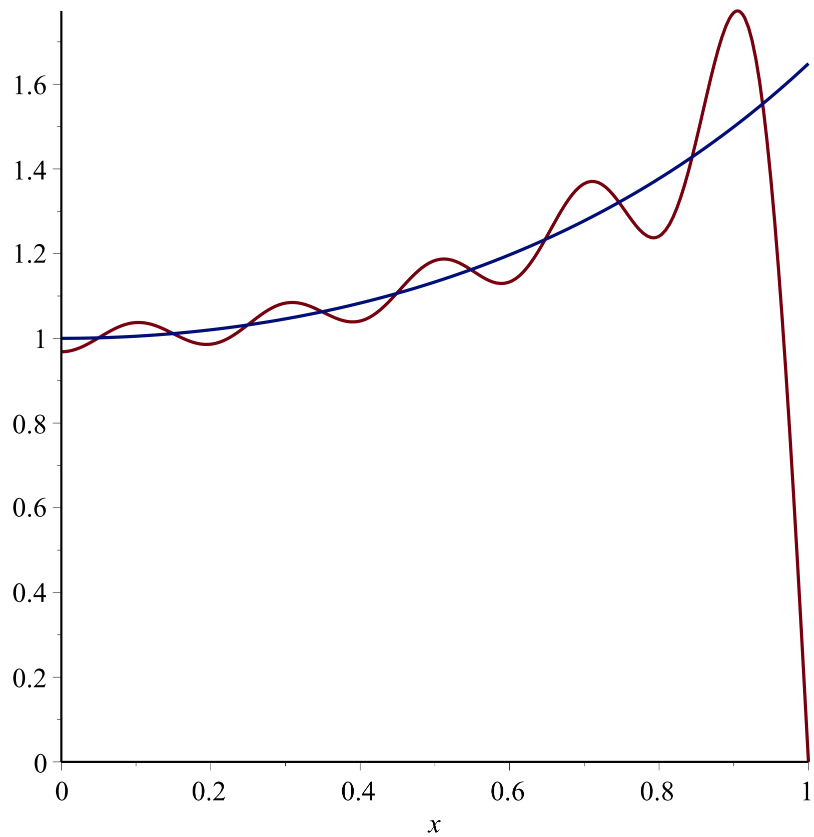
Now we approximate $f(x) = e^{x^2/2}$ by plugging our eigenvalues and eigenfunctions into the summation formula:

> $approx := sum((C[k]) \cdot fun[k], k = 1..10)$

$$approx := \frac{4\, e^{\frac{1}{2}x^2} \cos\left(\frac{1}{2} x \pi\right)}{\pi} - \frac{4}{3} \frac{e^{\frac{1}{2}x^2} \cos\left(\frac{3}{2} x \pi\right)}{\pi} + \frac{4}{5} \frac{e^{\frac{1}{2}x^2} \cos\left(\frac{5}{2} x \pi\right)}{\pi} - \frac{4}{7} \frac{e^{\frac{1}{2}x^2} \cos\left(\frac{7}{2} x \pi\right)}{\pi} + \frac{4}{9} \frac{e^{\frac{1}{2}x^2} \cos\left(\frac{9}{2} x \pi\right)}{\pi} - \frac{4}{11} \frac{e^{\frac{1}{2}x^2} \cos\left(\frac{11}{2} x \pi\right)}{\pi} + \frac{4}{13} \frac{e^{\frac{1}{2}x^2} \cos\left(\frac{13}{2} x \pi\right)}{\pi} - \frac{4}{15} \frac{e^{\frac{1}{2}x^2} \cos\left(\frac{15}{2} x \pi\right)}{\pi} + \frac{4}{17} \frac{e^{\frac{1}{2}x^2} \cos\left(\frac{17}{2} x \pi\right)}{\pi} - \frac{4}{19} \frac{e^{\frac{1}{2}x^2} \cos\left(\frac{19}{2} x \pi\right)}{\pi} \quad \textbf{(10)}$$

Let us plot the approximation and the function to be approximated on the same axes

> $plot\left(\left\{\exp\left(\frac{x^2}{2}\right), approx\right\}, x = 0..1\right)$



We can see that this is a pretty good approximation everywhere except near x = 1 where the function is discontinuous.