Question 5 (12 points)

Consider the vector field $\mathbf{F} = (3x + y^2)\mathbf{i} + e^z\mathbf{j} + xz\mathbf{k}$. Let S_1 be the part of the sphere $x^2 + y^2 + z^2 = 9$ above the xy-plane, and let S_2 be the part of the paraboloid $z = x^2 + y^2 - 9$ below the xy-plane, both with the upward orientation. Compute the difference

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$$

between the flux of $\mathbf{F}$ across S_1 and the flux of $\mathbf{F}$ across S_2.

Answer:

Although it is possible to compute the fluxes directly, a simpler approach is to apply the divergence theorem to the volumes enclosed by the curves and the xy-plane:

$$\oiint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \oiiint_V \mathrm{div}(\mathbf{F})\,\mathrm{d}V$$

First we calculate the divergence:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 3 + x$$

To take advantage of the symmetry of the region S_1, we rewrite this in spherical coordinates:

$$\nabla \cdot \mathbf{F} = 3 + \rho\cos\phi\cos\theta$$

Integrating this over the volume of the half-sphere, we have

$$D_1 = \int_0^3 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \big(3 + \rho\cos\phi\cos\theta\big)\rho^2 \sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}\rho$$

$$= 3\int_0^3 \rho^2\,\mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\frac{\pi}{2}} \sin\phi\,\mathrm{d}\phi$$

$$+ \int_0^3 \rho^3\,\mathrm{d}\rho \int_0^{2\pi} \cos\theta\,\mathrm{d}\theta \int_0^{\frac{\pi}{2}} \sin\phi\cos\phi\,\mathrm{d}\phi$$

The second term vanishes due to the periodicity of the θ integral. The first can be computed with straightforward techniques. Its value is $\boxed{54\pi}$.

Meanwhile, the integral over the paraboloidal region below the xy-plane can most easily be computed in cylindrical coordinates.

$$D_2 = \int_0^{2\pi} \int_0^3 \int_0^{r^2-9} (3 + r\cos\theta) r \, dz \, dr \, d\theta$$

$$= 3 \int_0^{2\pi} d\theta \int_0^3 r \, dr \int_0^{r^2-9} dz \; + \int_0^{2\pi} \cos\theta \, d\theta \int_0^3 r^2 \, dr \int_0^{r^2-9} dz$$

$$= \boxed{-\frac{243\pi}{2}}$$

Where again the second integral has vanished due to periodicity.

Finally, we are asked to compute the difference in the fluxes, so we compute

$$\int\int_{S_1} \mathbf{F} \cdot d\mathbf{S} - \int\int_{S_2} \mathbf{F} \cdot d\mathbf{S} = 54\pi - \left(\frac{243\pi}{2}\right) = \boxed{\frac{351\pi}{2}}$$