Question 5 (12 points)

Consider the vector field $\mathbf{F} = (3x + y^2)\mathbf{i} + e^z\mathbf{j} + xz\mathbf{k}$. Let S_1 be the part of the sphere $x^2 + y^2 + z^2 = 9$ above the xy-plane, and let S_2 be the part of the paraboloid $z = x^2 + y^2 - 9$ below the xy-plane, both with the upward orientation. Compute the difference

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$$

between the flux of $\mathbf{F}$ across S_1 and the flux of $\mathbf{F}$ across S_2.

Solution:

Let's calculate the flux through S_1. First, in order to take advantage of the spherical symmetry, we rewrite $\mathbf{F}$ in terms of spherical coordinates

$$\mathbf{F} = \langle 3\rho\cos\theta\sin\phi + \rho^2\sin^2\theta\sin^2\phi, e^{\rho\sin\phi}, \rho^2\cos\theta\cos\phi\sin\phi\rangle$$

Meanwhile the area element $\mathrm{d}\mathbf{S}$ can be written as

$$\mathrm{d}\mathbf{S} = \rho^2\sin\phi\mathrm{d}\phi\mathrm{d}\theta\hat{\rho}$$

where the unit vector $\hat{\rho}$ can be written

$$\begin{aligned}\hat{\rho} &= \frac{1}{\rho}\langle x, y, z\rangle \\ &= \frac{1}{\rho}\langle \rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi\rangle \\ &= \langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle\end{aligned}$$

Substituting these expressions into the flux expression, we have

$$\begin{aligned}\int\int_{S_1} \mathbf{F}\cdot\mathrm{d}\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi/2} \left(3\rho\cos\theta\sin\phi + \rho^2\sin^2\theta\sin\phi\right)\left(\rho^2\cos\theta\sin^2\phi\right)\mathrm{d}\phi\mathrm{d}\theta \\ &+ \int_0^{2\pi}\int_0^{\pi/2} \left(e^{\rho\sin\phi}\right)\left(\rho^2\sin\theta\sin^2\phi\right)\mathrm{d}\phi\mathrm{d}\theta \\ &+ \int_0^{2\pi}\int_0^{\pi/2} \left(\rho^2\cos\theta\cos\phi\sin\phi\right)\left(\rho^2\cos\phi\sin\phi\right)\mathrm{d}\phi\mathrm{d}\theta\end{aligned}$$

After a bit of factoring, we have

$$
\begin{aligned}
&= 3\rho^3 \int_0^{2\pi} \cos^2\theta \mathrm{d}\theta \int_0^{\frac{\pi}{2}} \sin^3\phi \mathrm{d}\phi + \rho^4 \int_0^{2\pi} \cos\theta \sin^2\theta \mathrm{d}\theta \int_0^{\frac{\pi}{2}} \sin^3\phi \mathrm{d}\phi \\
&\quad + \rho^2 \int_0^{2\pi} \sin\theta \mathrm{d}\theta \int_0^{\frac{\pi}{2}} e^{\rho\sin\phi} \sin^2\phi \mathrm{d}\phi \\
&\quad + \rho^4 \int_0^{2\pi} \cos\theta \mathrm{d}\theta \int_0^{\frac{\pi}{2}} \cos^2\phi \sin^2\phi \mathrm{d}\phi
\end{aligned}
$$

All but the first of these terms are zero due to periodicity of the θ integrals. The first can be computed using standard techniques

$$= 3\rho^3 \int_0^{2\pi} \frac{1+\cos(2\theta)}{2} \mathrm{d}\theta \int_0^{\frac{\pi}{2}} \left(\sin\phi - \sin\phi\cos^2\phi\right)\mathrm{d}\phi$$

$$= 3\rho^3 \left[\theta + \cancelto{0}{\frac{1}{2}\sin(2\theta)}\right]_0^{2\phi} \left(1 + \int_1^0 u^2 \mathrm{d}u\right) \qquad \text{(using the substitution } u = \cos\phi\text{)}$$

$$= \frac{3\rho^3}{2}(2\pi)\frac{2}{3}$$

$$= \boxed{54\pi}$$

where in the last step we have substituted the radius of the sphere $\rho = 3$.

Next we calculate the flux through the paraboloidal surface S_2.

Here is it is helpful to use the following equation (Formula 16.10 in *Stewart* 7th ed.):

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

where $z = g(x, y) = x^2 + y^2 - 9$

and P, Q, and R and the x-, y-, and z-components of $\mathbf{F}$, respectively.

Accordingly, we have

$$\int\int_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int\int \left[-\left(3x + y^2\right)(2x) - e^z(2y) + xz\right] \mathrm{d}A.$$

After converting to cylindrical coordinates, $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$, and simplifying, this becomes

$$= \int_0^{2\pi} \int_0^3 \left(-6r^2\cos^2\theta - 2r^3\cos\theta\sin^2\theta - 2re^{r^2-9}\sin\theta + r\cos\theta(r^2-9)\right) r\,\mathrm{d}r\,\mathrm{d}\theta.$$

As before, all but the first term will vanish due to symmetry, and the resulting integral is

$$= \boxed{-\frac{243\pi}{2}}$$

Finally, we can compute the difference in the fluxes

$$\int\int_{S_1} \mathbf{F}\cdot\mathrm{d}\mathbf{S} - \int\int_{S_2} \mathbf{F}\cdot\mathrm{d}\mathbf{S} = 54\pi - \left(-\frac{243\pi}{2}\right) = \boxed{\frac{351\pi}{2}}$$