Q: Let C be the rim of the rectangle with vertices (0, 0), (2, 0), (2, 1), and (0, 1). Use Green's formula to compute the integral

$$\int_C \left(x^2y^2\mathrm{d}x + x^3y\mathrm{d}y\right)$$

A: Green's theorem relates a path integral along a curve C to an area integral on a region A.

$$\oint_C (L\mathrm{d}x + M\mathrm{d}y) = \iint_A \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y$$

The origin of this formula is a bit murky, but it is a special case of the Kelvin-Stokes theorem which states that the path integral of a vector field along a closed curve is equal to the integral of the curl over the enclosed area.

Applying this formula to our rectangle is straightforward. Identifying $M = x^3y$ and $L = x^2y^2$ and computing the partial derivatives, we have

$$\begin{aligned}\iint_A \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y &= \int_0^1 \int_0^2 \left(3x^2y - 2x^2y\right) \mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \left(\left[x^3y - \frac{2}{3}x^3y\right]_0^2\right) \mathrm{d}y \\ &= \int_0^1 \left(8y - \frac{16}{3}y\right) \mathrm{d}y \\ &= \left[4y^2 - \frac{8}{3}y^2\right]_0^1 = 4 - \frac{8}{3} = \boxed{\frac{4}{3}}\end{aligned}$$

Note that the order of integration does not matter.

We can confirm this result by computing the path integral along the four edges of the rectangle:

$$(0,0) \longrightarrow (2,0) \qquad (2,0) \longrightarrow (2,1) \qquad (2,1) \longrightarrow (0,1) \qquad (0,1) \longrightarrow (0,0)$$

$$\int_0^2 x^2(0^2)\mathrm{d}x \quad + \quad \int_0^1 (2)^3y\mathrm{d}y \quad + \quad \int_2^0 x^2(1)^2\mathrm{d}x \quad + \quad \int_1^0 (0)^3y\mathrm{d}y$$

$$= \quad 0 \quad + \quad 4 \quad - \quad \frac{8}{3} \quad + \quad 0 \quad = \quad \boxed{\frac{4}{3}}$$