Q: Solve the second-order linear homogeneous ordinary differential equation

$$xy'' - (12x^2+1)y' + 36x^3y = 0$$

A: We begin by guessing the general form of a solution. Since the coefficient functions $\left(36x^3, -12x^2-1, x\right)$ are polynomials whose order decreases as the number of derivatives increases, we guess that $y(x)$ is a function with polynomials coefficients that *increase* each time we take a derivative.

One such function with this property is

$$y(x) = e^{kx^2}$$

Computing the derivatives

$$y'(x) = 2kxe^{kx^2}$$

$$y'' = 2k\left(2kx^2+1\right)e^{kx^2}$$

and substituting into the ODE, we have

$$2kx\left(2kx^2+1\right)e^{kx^2} - 2kx(12x^2+1)e^{kx^2} + 36x^3e^{kx^2} = 0$$

which simplifies to

$$4k^2x^3 + 2kx - 24kx^3 - 2kx + 36x^3 = 0$$

$$4x^3\left(k^2-6k+9\right) = 0$$

This equation is satisfied if $k=3$, so therefore

$$y_1(x) = e^{3x^2}$$

is a solution to the differential equation. We can find another independent solution by using the formula

$$y_2 = y_1\int\frac{1}{y_1^2}e^{-\int P(x)\mathrm{d}x}\mathrm{d}x$$

where

$$P(x) = -\frac{12x^2+1}{x}$$

is the coefficient function of the $y'(x)$ term of the ODE when both sides are divided by x so that the

$y''(x)$ coefficient is 1. Completing the calculation, we have

$$y_2 = e^{3x^2} \int e^{-6x^2} e^{\int \left(12x + \frac{1}{x}\right) \mathrm{d}x} \mathrm{d}x = e^{3x^2} \int e^{-6x^2} e^{6x^2 + \ln|x|} \mathrm{d}x = e^{3x^2} \int x \mathrm{d}x = \frac{x^2}{2} e^{3x^2}$$

Therefore the set

$$B = \left\{ e^{3x^2}, x^2 e^{3x^2} \right\}$$

is a basis for the solution space and the general solution can be written

$$\boxed{y(x) = c_1 e^{3x^2} + c_2 x^2 e^{3x^2}}$$