Q: Find an integrating factor as a function of y only that reduces the following differential equation to an exact one.

$$\left(2xy^2 + y\right)\mathrm{d}x - x\mathrm{d}y = 0 \tag{1}$$

A: Let $M = \frac{\partial V}{\partial x} = 2xy^2 + y$ and $N = \frac{\partial V}{\partial y} = -x$ where $V(x, y) = \text{ constant}$. To test whether the original equation is exact, we compute

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = (4xy + 1) - (-1) = 4xy + 2 \tag{2}$$

Since this is non-zero, that means our differential is inexact. We multiply both sides of equation (1) by a new function $\mu(y)$

$$(2xy^2 + y)\mu(y)\mathrm{d}x - x\mu(y)\mathrm{d}y \tag{3}$$

and recompute our test function (2), setting the right hand side equal to zero.

$$\frac{\partial(\mu M)}{\partial y} - \frac{\partial(\mu N)}{\partial x} = 0 \tag{4}$$

This becomes

$$\mu\frac{M}{\partial y} + M\frac{\mathrm{d}\mu}{\mathrm{d}y} - \mu\frac{\partial N}{\partial x} = 0 \tag{5}$$

(Notice that the $\partial\mu/\partial x$ term is missing since the problem assumes μ is independent of x.) Continuing with our computation, we get

$$(4xy + 1)\mu + (2xy^2 + y)\frac{\mathrm{d}\mu}{\mathrm{d}y} - \mu(-1) = 0 \tag{6}$$

This differential equation is separable. Simplifying and solving for μ gives

$$(2xy^2 + y)\frac{\mathrm{d}\mu}{\mathrm{d}y} = -(4xy + 2)\mu \tag{7}$$

$$\frac{\mathrm{d}\mu}{\mu} = -\frac{4xy + 2}{2xy^2 + y}\mathrm{d}y$$

$$\frac{\mathrm{d}\mu}{\mu} = -\frac{2}{y}\mathrm{d}y$$

$$\ln\mu = -2\ln y$$

$$\mu = e^{-2\ln y}$$

$$\boxed{\mu = y^{-2}} \tag{8}$$

Supplement 1: Solving the equation
Substituting our solution (8) into equation (3) gives

$$\left(2x + \frac{1}{y}\right)\mathrm{d}x - \frac{x}{y^2}\mathrm{d}y \tag{9}$$

We can test that this is indeed exact by computing

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0 - \frac{1}{y^2} + \frac{1}{y^2} = 0 \tag{10}$$

Since the equation is exact, we can write (9) as the total differential

$$dV = M\mathrm{d}x + N\mathrm{d}y \tag{11}$$

of some potential function $V(x, y) = \mathrm{constant}$ where

$$M = \frac{\partial V}{\partial x} = 2x + \frac{1}{y} \tag{12}$$

and

$$N = -\frac{x}{y^2} = \frac{\partial V}{\partial y} \tag{13}$$

To compute V, let us integrate (13)

$$V(x, y) = \int \frac{\partial V}{\partial y}\mathrm{d}y = -\int \frac{x}{y^2}\mathrm{d}y = \frac{x}{y} + \phi(x) + C \tag{14}$$

Here $\phi(x)$ is some unknown function of x. To find ϕ, we differentiate both sides of (14) with respect to x and set the result equal to (12).

$$\frac{1}{y} + \phi'(x) = 2x + \frac{1}{y} \tag{15}$$

From this, we can see that $\phi'(x) = 2x$ and therefore $\phi(x) = x^2$.

Finally, we substitute ϕ back into (12) and solve for y.

$$\frac{x}{y} + x^2 + C = \mathrm{constant}$$

$$\boxed{y(x) = -\frac{x}{x^2 + C}} \tag{16}$$

This is the general solution to the differential equation (1).

Supplement 2: An easier way to solve

It is not necessary to make the differential exact in order to solve it. Rearranging equation (1), we compute

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy^2 + y}{x} = 2y^2 + \frac{y}{x} \tag{17}$$

Through inspired guesswork, we introduce a new variable v such that

$$v = \frac{y}{x} \tag{18}$$

Accordingly, we see that

$$y = vx \tag{19}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x} \tag{20}$$

We can eliminate the variable y by substituting (19) and (20) into (17).

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 2v^2x^2 + v$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = 2v^2x \tag{21}$$

This equation is easily solved by separation of variables.

$$\frac{\mathrm{d}v}{v^2} = 2x\mathrm{d}x \tag{22}$$

$$-\frac{1}{v} = x^2 + C$$

$$v = -\frac{1}{x^2 + C} \tag{23}$$

We complete the problem by substituting (23) into (19).

$$\boxed{y = -\frac{x}{x^2 + C}} \tag{24}$$