Question: Use an *integrating factor* to solve the following first-order differential equation

$$x^3y' - x^2y = 3x \tag{1}$$

Solution: The goal is to manipulate the left side of (1) to become the derivative of a product. First, let's simplify by dividing both sides by x^3.

$$y' - \frac{1}{x}y = \frac{3}{x^2} \tag{2}$$

Next we multiply both sides by the (unknown) integrating factor $\mu(x)$

$$\mu y' + -\frac{\mu}{x}y = \frac{3}{x^2}\mu \tag{3}$$

By definition, $\mu(x)$ has the property that the left side of (3) will become

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu y' + \mu' y \tag{4}$$

Comparing the RHS of (4) with the LHS of (3) with the last, we see that

$$\mu' = -\frac{\mu}{x} \tag{5}$$

This is a separable differential equation that can be solved as follows:

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = -\frac{\mu}{x}$$

$$\frac{\mathrm{d}\mu}{\mu} = -\frac{\mathrm{d}x}{x}$$

$$\ln\mu = -\ln x$$

$$\mu = e^{-\ln x}$$

$$\mu = \left(e^{\ln x}\right)^{-1}$$

$$\mu = x^{-1} \tag{6}$$

(Note that it is okay to ignore the constant of integration ($+C$) since we only need *a* solution—*not* the general solution.) Now that we have solved for $\mu(x)$, we can, according to equation (4), replace the LHS of (3) with $\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)$. This results in

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}y\right) = \frac{3}{x^3} \tag{7}$$

This can be easily solved by integrating both sides:

$$y = x \int \frac{3}{x^3}\,\mathrm{d}x$$

$$= 3x\left(-\frac{1}{2}x^{-2} + C\right)$$

$$\boxed{y = -\frac{3}{2x} + Cx} \tag{8}$$

Final note: In the future, to save time solving for $\mu(x)$, we can memorize that that if the differential equation is written in the form

$$y' + p(x)y = q(x) \tag{9}$$

then the integrating factor can be written

$$\boxed{\mu(x) = e^{-\int p(x)\mathrm{d}x}} \tag{10}$$

and the general solution will be

$$\boxed{y = e^{\int p(x)\mathrm{d}x} \int q(x)e^{-\int p(x)\mathrm{d}x}\mathrm{d}x} \tag{11}$$