Question 3: A small lake, which is initially clean and has an initial volume of 40,000 m^3 has two creeks flowing into it (creek A and creek B) and one creek flowing out (creek C). Each day, 800 m^3 of water flows into the lake from creek A, 200 m^3 per day flow into the lake from creek B, and 1000 m^3 flows out of the lake via creek C. At time t = 0, the water flowing into the lake from creek A becomes contaminated with road salt at a concentration of 10 kg per 400 m^3. Suppose that the water in the lake is well mixed so that the concentration of salt at any given time is constant. To complicated the matter, suppose also that at time t = 0, someone begins dumping trash into the lake at a rate of 50 m^3 per day, which settles to the bottom of the lake, reducing the volume by 50 m^3 per day. To adjust for the incoming trash, the rate that water flows out via creek C increases to 1300 m^3 per day and the banks of the lake do not overflow. Find the amount of salt in the lake at time t, where t is in days.

Solution: To calculate the amount of salt in the lake at a given time, we write

$$S' = S'_{\text{in}} - S'_{\text{out}} \tag{1}$$

In this equation,

$$S'_{\text{in}} = C_{\text{A}} \cdot V'_{\text{in,A}} \tag{2}$$

where C_{A} = 0.025 kg/m^3 is the salt concentration in stream A and $V'_{\text{in,A}} = 800$ m^3/day is the volume of water entering the lake per day from stream A. Meanwhile, S'_{out} is given by

$$S'_{\text{out}} = \frac{S(t)}{V(t)} V'_{\text{out}} \tag{3}$$

where $S(t)/V(t)$ is the current salt concentration in the lake and V'_{out}, has increased to 1300 m^3/day. Since the volume is decreasing at a rate of 50 m^3/day, we can write

$$V(t) = 40000 - 50t \tag{4}$$

Plugging (4), (5), and (6) equation (3) gives

$$S' = 20 - \frac{1300S}{40000 - 50t} \tag{5}$$

Which can be rewritten as

$$S' + \frac{26}{800 - t} S = 20 \tag{6}$$

This is a first order non-separable differential equation. To solve it, we first compute the integrating factor, $\mu(t)$.

$$\mu(t) = e^{\int \frac{26}{800-t} \mathrm{d}t}$$

$$\mu(t) = e^{-26 \ln(800-t)}$$

$$\mu(t) = e^{\ln(800-t)^{-26}}$$

$$\mu(t) = (800 - t)^{-26}$$

$$\mu(t) = \frac{1}{(800 - t)^{26}} \tag{7}$$

Notice that when we multiply both sides of equation (8) by $\mu(t)$, the left side becomes (according to the quotient rule) the derivative of a single function $\mu(t)S(t)$.

$$\frac{1}{(800 - t)^{26}} S'(t) + \frac{26}{(800 - t)^{27}} S = \frac{20}{(800 - t)^{26}} \tag{8}$$

Therefore, we can write,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{S(t)}{(800 - t)^{26}} \right) = \frac{20}{(800 - t)^{26}} \tag{9}$$

To solve for $S(t)$, we integrate both sides and multiply $(800 - t)^{26}$

$$S(t) = (800 - t)^{26} \int \frac{20 \mathrm{d}t}{(800 - t)^{26}} \tag{10}$$

After making the substitution

$$u = 800 - t \quad \longrightarrow \quad \mathrm{d}t = -\mathrm{d}u \tag{11}$$

our equation becomes

$$S(t) = -20(800 - t)^{26} \int u^{-26} \mathrm{d}u \tag{12}$$

$$S(t) = 20(800 - t)^{26} \left(\frac{(800 - t)^{-25}}{25} + C \right) \tag{13}$$

$$S(t) = \frac{4}{5}(800 - t) + C(800 - t)^{26} \tag{14}$$

This is the general solution to the differential equation. Plugging in the initial condition $S(0) = 0$ gives

$$0 = \frac{4}{5}(800) + C(800)^{26} \tag{15}$$

$$C = -\frac{4}{5}(800)^{-25} \tag{16}$$

Substituting this back into (16) results in, after a bit of simplification, a final answer of

$$\boxed{S(t) = \frac{4}{5}(800 - t) \left[1 - \left(\frac{800 - t}{800} \right)^{25} \right]} \tag{17}$$

Plotting S(t) gives the following graph. We can see that at first, the salt level rises quickly, reaching a maximum at about 100 days. After then the concentration is high enough that the amount of salt exiting the lake is greater than the amount entering. At 800 days the lake is drained entirely, after which the function is no longer physically meaningful.