Q: Solve $y''(t) - 3y'(t) + 2y(t) = 1$ with initial conditions $y(0) = 2, y'(0) = 1$

A: Take the Laplace transform of both sides

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) - 3sY(s) + 3y(0) + 2Y(s) &= \frac{1}{s} \\ s^2Y(s) - 2s - 1 - 3sY(s) + 6 + 2Y(s) &= \frac{1}{s} \\ (s^2 - 3s + 2)Y(s) &= 2s - 5 + \frac{1}{s} \\ (s-1)(s-2)Y(s) &= \frac{2s^2 - 5s + 1}{s} \end{aligned}$$

$$Y(s) = \frac{2s^2 - 5s + 1}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$$

The partial fraction decomposition is completed as follows:

$$2s^2 - 5s + 1 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$$

$$2s^2 - 5s + 1 = (A + B + C)s^2 + (-3A - 2B - C)s + 2A$$

$$\begin{aligned} A + B + C &= 2 \\ -3A - 2B - C &= -5 \\ 2A &= 1 \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2} \\ B &= 2 \\ C &= -\frac{1}{2} \end{aligned}$$

$$Y(s) = \frac{\frac{1}{2}}{s} + \frac{2}{s-1} - \frac{\frac{1}{2}}{s-2}$$

The inverse-Laplace transform of this is

$$\boxed{y(t) = \frac{1}{2} + 2e^t - \frac{1}{2}e^{2t}}$$

Q: Solve $y''(t) - 3y'(t) + 2y(t) = 1$ with initial conditions $y(0) = 2, y'(0) = 1$

A: Start by solving characteristic polynomial of the homogeneous system:

$\lambda^2 - 3\lambda + 2 = 0 \longrightarrow \lambda \in \{1, 2\}$

This gives the complimentary solutions

$y_c = Ae^t + Be^{2t}$

We assume a particular solution of the form

$y_p = y_c + C$ where C is an unknown constant to match the RHS of the original ODE

Substituting this into the ODE gives

$0 + \frac{C}{2} = 1 \longrightarrow C = \frac{1}{2}$

Now that we know the general solution

$y(t) = Ae^t + Be^{2t} + \frac{1}{2}$

$y'(t) = Ae^t + 2Be^{2t}$

we apply the initial conditions

$2 = A + B + \frac{1}{2}$

$1 = A + 2B$

which has solution

$A = 2$
$B = -\frac{1}{2}$

Therefore we have

$$\boxed{y(t) = 2e^t - \frac{1}{2}e^{2t} + \frac{1}{2}}$$

This matches the solution from the previous page.