Problem: Solve the ODE

$$y'' + 2y(y')^3 = 0 \tag{1}$$

Solution:
Start by introducing a new function $v(y)$ such that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v \tag{2}$$

This implies

$$y'' = \frac{\mathrm{d}v}{\mathrm{d}x} \tag{3}$$

However, since x does not appear in the original ODE, it is undesirable to introduce a new variable. Accordingly we rewrite equation (3) using the chain rule.

$$y'' = \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = v\frac{\mathrm{d}v}{\mathrm{d}y} \tag{4}$$

Substituting (2) and (4) into (1), we find

$$v\frac{\mathrm{d}v}{\mathrm{d}y} + 2yv^3 = 0 \tag{5}$$

Equation (5) is a separable first order ODE, so we can solve in the usual manner:

$$\frac{\mathrm{d}v}{v^2} = -2y\mathrm{d}y \tag{6}$$

$$-\frac{1}{v} = -y^2 + C \tag{7}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v = \frac{1}{y^2 + C} \tag{8}$$

We can now compute $y(x)$ by separating x and y and integrating both sides.

$$\int (y^2 + C)\mathrm{d}y = \int \mathrm{d}x \tag{9}$$

$$\boxed{\frac{1}{2}y^3 + Cy + D = x} \tag{10}$$

This is a cubic equation, so it can in principle be solved for $y(x)$; however the process is cumbersome, so I will not pursue it.