Problem: Solve the ODE

$$2yy'' = y^2 + (y')^2 \tag{1}$$

Solution: This is a second order differential equation, so we begin by introducing a new function $v(y)$ such that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v. \tag{2}$$

This implies

$$y'' = \frac{\mathrm{d}v}{\mathrm{d}x}. \tag{3}$$

However, since the independent variable x does not appear in the original ODE, it is undesirable to introduce an explicit x dependency. Accordingly we rewrite equation (3) using the chain rule:

$$y'' = \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = v\frac{\mathrm{d}v}{\mathrm{d}y} \tag{4}$$

Substituting (2) and (4) into (1), we find

$$2yv\frac{\mathrm{d}v}{\mathrm{d}y} = y^2 + v^2. \tag{5}$$

Dividing both sides by yv converts our equation into

$$2\frac{\mathrm{d}v}{\mathrm{d}y} = \frac{y}{v} + \frac{v}{y}. \tag{6}$$

This equation is homogeneous in the sense that we can eliminate v by introducing a new function w such that

$$w = \frac{v}{y}. \tag{7}$$

This means that

$$v = wy \tag{8}$$

and (applying the product rule),

$$\frac{\mathrm{d}v}{\mathrm{d}y} = y\frac{\mathrm{d}w}{\mathrm{d}y} + w. \tag{9}$$

Substituting (8) and (9) into (6) yields

$$2y\frac{\mathrm{d}w}{\mathrm{d}y} + 2w = \frac{1}{w} + w \tag{10}$$

which is a separable differential equation. After a bit of algebraic rearrangement, (10) turns into

$$\frac{-2w\mathrm{d}w}{1 - w^2} = -\frac{\mathrm{d}y}{y}. \tag{11}$$

Integrating both sides, we have

$$\ln(1-w^2) = -\ln y + \ln A \tag{12}$$

where A is an arbitrary constant. This simplifies to

$$1 - w^2 = \frac{A}{y}. \tag{13}$$

Now we work backward. Substituting (7) into (13), and multiplying both sides by y^2 yields

$$y^2 - v^2 = Ay. \tag{14}$$

Replacing v with $\mathrm{d}y/\mathrm{d}x$, according to equation (2), we have another separable differential equation which can be rearranged as follows:

$$y^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = Ay$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{y^2 - Ay}$$

$$\frac{\mathrm{d}y}{\sqrt{y^2 - Ay}} = \mathrm{d}x. \tag{15}$$

Making a change of variables

$$u = \sqrt{\frac{y}{A}} \quad \longrightarrow \quad y = Au^2 \quad \longrightarrow \quad \mathrm{d}y = 2A\mathrm{d}u \tag{16}$$

and integrating, we get

$$\int \frac{2\mathrm{d}u}{\sqrt{u^2-1}} = \int \mathrm{d}x. \tag{17}$$

Using the fact that $\int \frac{\mathrm{d}u}{\sqrt{u^2-1}} = \cosh^{-1} u$, this becomes

$$2\cosh^{-1}\sqrt{\frac{y}{A}} = x + C \tag{18}$$

Finally, by solving for y, we arrive at the general solution:

$$\boxed{y = A\cosh^2\left(\frac{x+C}{2}\right)} \tag{19}$$

Side note:

You may have noticed through trial and error that $y = e^x$ and $y = e^{-x}$ are also solutions to (1). We can reconcile this fact with our general solution by noting that e^x and e^{-x} are the limits of (19) as A approaches e^{-C} and while C approaches ∞ and $-\infty$, respectively.

Given that e^x and e^{-x} are solutions, you might be tempted to guess that

$$y = Ae^x + Be^{-x}$$

is also a solution, but this turns out not to be the case. If (1) were a *linear* ODE, this would be true, but it is not.