Q: Find a one-parameter family of solutions for the following first-order differential equation:

$$\left(x^3 + y^2\sqrt{x^2+y^2}\right)\mathrm{d}x - \left(xy\sqrt{x^2+y^2}\right)\mathrm{d}y = 0 \tag{1}$$

A: We start by solving this for $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^3 + y^2\sqrt{x^2+y^2}}{xy\sqrt{x^2+y^2}} \\ &= \frac{x^2}{y\sqrt{x^2+y^2}} + \frac{y}{x}\end{aligned} \tag{2}$$

Let us rewrite y in terms of a new function v

$$y = vx \tag{3}$$

Differentiating each side of (4) using the product rule gives'

$$y' = v'x + v \tag{4}$$

Substituting (3) and (4) into (2) results in

$$v'x + v = \frac{x^2}{vx\sqrt{x^2+v^2x^2}} + v \tag{ 5}$$

This simplifies nicely to

$$v'x = \frac{1}{\sqrt{1+v^2}} \tag{6}$$

Which is a separable differential equation

$$\sqrt{1+v^2}\,\mathrm{d}v = \frac{\mathrm{d}x}{x} \tag{7}$$

The left side is a famous integral which can be found in tables. Integrating both sides results in

$$\frac{1}{2}v\sqrt{v^2+1} + \frac{1}{2}\sinh^{-1}(v) = \ln|x| + C \tag{8}$$

Or

$$\boxed{\frac{1}{2}\left(\frac{y}{x}\right)\sqrt{1+\left(\frac{y}{x}\right)} + \frac{1}{2}\sinh^{-1}\left(\frac{y}{x}\right) = \ln|x| + C}$$