Problem: Find the solution to the initial value problem:

$$y'' + 3y' + 2y = tH(t-2), \qquad y(0) = 1, \qquad y'(0) = 0 \tag{1}$$

Solution: We begin by taking the Laplace transform of both sides of (1), using the usual rules for the Laplace transforms of derivatives and the fact that $\mathcal{L}\{tf(t)\} = -F'(\lambda)$

$$\lambda^2 Y(\lambda) - \lambda y(0) - y'(0) + 3\lambda Y(\lambda) - 3y(0) + 2Y(\lambda) = -\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{e^{-2\lambda}}{\lambda} \tag{2}$$

Next we substitute our initial conditions into (2) and compute the derivative on the right-hand side

$$\lambda^2 Y(\lambda) - \lambda + 3\lambda Y(\lambda) - 3 + 2Y(\lambda) = \frac{e^{-2\lambda}(2\lambda+1)}{\lambda^2} \tag{3}$$

Solving (3) for $Y(\lambda)$ and factoring the denominators yields

$$Y(\lambda) = \frac{\lambda+3}{(\lambda+1)(\lambda+2)} + \frac{e^{-2\lambda}(2\lambda+1)}{\lambda^2(\lambda+1)(\lambda+2)} \tag{4}$$

At this point, we have technically completed the problem, but it is usually desirable to rewrite the solution in terms of the original variable t. The inverse-Laplace transform of the first fraction will yield the homogeneous solution $y_h(t)$ which solves $y'' + 3y' + 2 = 0$ subject to the initial conditions. Meanwhile, the second fraction will yield the particular solution $y_p(t)$, which is determined by the right-hand side of equation (1). Since $\mathcal{L}^{-1}\{e^{-a\lambda}F(\lambda)\} = f(t-a)H(t-a)$, it is convenient to write the general solution as

$$y(t) = y_h(t) + y_p(t-2)H(t-2) \tag{5}$$

where $y_p(t)$ is given by

$$y_p(t) = \mathcal{L}^{-1}\left\{\frac{2\lambda+1}{(\lambda+1)(\lambda+2)}\right\} \tag{6}$$

Our strategy is to compute the inverse-Laplace transforms using the residue theorem

$$y(t) = \mathcal{L}^{-1}\{Y(\lambda)\} = \sum_{\text{poles}} \text{Res}\left(Y(\lambda)e^{\lambda t}\right) \tag{7}$$

where the sum is over all the set of poles (i.e. roots of the denominator) of $Y(\lambda)$. Generally speaking, the residue of a function $G(\lambda)$ with respect to a pole λ_i is given by given by

$$\text{Res}\left(G(\lambda), \lambda_i\right) = \frac{1}{(n-1)!}\lim_{\lambda \to \lambda_i}\frac{\mathrm{d}^{n-1}}{\mathrm{d}\lambda^{n-1}}\left(\lambda - \lambda_i\right)^n G(\lambda) \tag{8}$$

In the particular case where we have a pole of order $n = 1$, this simplifies to

$$\text{Res}\left(G(\lambda), \lambda_i\right) = \lim_{\lambda \to \lambda_i}(\lambda - \lambda_i)G(\lambda) \tag{9}$$

The effect is to evaluate the fraction at the pole after removing the corresponding factor from the denominator. Moreover, if $n = 2$, the residue becomes

$$\mathrm{Res}\,(G(\lambda), \lambda_i) = \lim_{\lambda \to \lambda_i} \frac{\mathrm{d}}{\mathrm{d}\lambda} (\lambda - \lambda_i)^2 G(\lambda) \tag{10}$$

Returning to equation (4), we observe that the $y_h(t)$-determining fraction in $Y(\lambda)$ has two poles:

$\lambda_1 = -1$ (1st order)
$\lambda_2 = -2$ (1st order)

and therefore its inverse Laplace transform is given by

$$y_h(t) = \mathrm{Res}\left(Y(\lambda)e^{\lambda t}, -1\right) + \mathrm{Res}\left(Y(\lambda)e^{\lambda t}, -2\right) \tag{11}$$

We now apply formulas (9) and (10) to see that

$$y_h(t) = \lim_{\lambda \to -1} \frac{e^{\lambda t}(\lambda + 3)}{\lambda + 2} + \lim_{\lambda \to -2} \frac{e^{\lambda t}(\lambda + 3)}{\lambda + 1} \tag{12}$$

resulting in

$$y_h(t) = 2e^{-t} - e^{-2t}$$

To compute $y_p(t)$, we take the inverse-Laplace transform in equation (6). This one has three poles:

$\lambda_1 = 0$ (2nd order)
$\lambda_2 = -1$ (1st order)
$\lambda_3 = -2$ (1st order)

Again applying formulas (9) and (10) we have

$$y_p(t) = \lim_{\lambda \to 0} \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{e^{\lambda t}(2\lambda + 1)}{(\lambda + 1)(\lambda + 2)} + \lim_{\lambda \to -1} \frac{e^{\lambda t}(2\lambda + 1)}{\lambda^2(\lambda + 2)} + \lim_{\lambda \to -2} \frac{e^{\lambda t}(2\lambda + 1)}{\lambda^2(\lambda + 1)} \tag{13}$$

After computing the derivative and taking the limit, this becomes

$$y_p(t) = \frac{t}{2} + \frac{1}{4} - e^{-t} + \frac{3}{4}e^{-2t} \tag{14}$$

Now that we have computed both $y_h(t)$ and $y_p(t)$, we can substitute these into (5) to get our general solution.

$$y(t) = \boxed{2e^{-t} - e^{-2t} + \left[\frac{t}{2} - \frac{3}{4} - e^{-(t-2)} + \frac{3}{4}e^{-2(t-2)}\right] H(t-2)} \tag{15}$$