Q: Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{2x} = \frac{x}{y^3} \qquad y(1) = 2 \tag{1}$$

A: The two fractions suggests that we try a substitution of the form

$$y = vx^n \tag{2}$$

where v is some unknown function of x and n is some unknown exponent, and hope that we can combine the fractions into a single term. Applying the product rule to (2), we have

$$y' = v'x^n + nvx^{n-1} \tag{3}$$

Substituting (2) and (3) into (1) gives us

$$v'x^n + nvx^{n-1} + \frac{vx^n}{2x} = \frac{x}{v^3x^{3n}}$$

or, after rewriting,

$$v'x^n = v^{-3}x^{1-3n} - \left(n + \frac{1}{2}\right)vx^{n-1} \tag{4}$$

If the powers of the x-terms on the right hand side are equal, we can simplify. Therefore we choose

$$1 - 3n = n - 1 \longrightarrow n = \frac{1}{2} \tag{5}$$

Substituting $n = \frac{1}{2}$ into (4) gives, after a bit of simplificiation

$$v'x^{1/2} = \left(v^{-3} - v\right)x^{-1/2} \tag{6}$$

This is a separable differential equation. We can rewrite it as

$$\frac{\mathrm{d}v}{v^{-3} - v} = \frac{\mathrm{d}x}{x} \tag{7}$$

The right hand side is just $\ln|x| + C$. The left can be rewritten as follows:

$$\int \frac{\mathrm{d}v}{v^{-3} - v} = \int \frac{v^3\,\mathrm{d}v}{1 - v^4} = -\frac{1}{4}\int \frac{4v^3\,\mathrm{d}v}{v^4 - 1} = -\frac{1}{4}\ln|v^4 - 1| \tag{8}$$

where we have used the fact that the numerator of the integrand is the derivative of the denominaor, i.e.

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + C \tag{9}$$

This turns Equation (7) into

$$-\frac{1}{4}\ln|v^4-1| = \ln|x| + C \tag{10}$$

Exponentiating both sides gives

$$(v^4-1)^{-1/4} = Ax \tag{11}$$

where A is a new arbitrary constant. (11) can be readily solved for v:

$$v = \pm\sqrt[4]{(Ax)^{-4}+1} \tag{12}$$

Or, using (2) and the fact that $n = \frac{1}{2}$,

$$y = \pm x^2\sqrt[4]{(Ax)^{-4}+1} \tag{13}$$

Finally, we can apply the initial condition $y(1) = 2$, to get $A^{-4} = 15$ so that our final answer becomes

$$\boxed{y = x^2\sqrt[4]{15x^{-4}+1}}$$