Q: Find the general solution to the following differential equation using the method of variation of parameters

$$(2x+1)(x+1)y'' + 2xy' - 2y = (2x+1)^2 \tag{1}$$

Given that $y_1 = x$ and $y_2 = \frac{1}{x+1}$ are linearly independent solutions to the corresponding homogeneous equation.

A: Dividing both sides of (1) by the leading coefficient, we have

$$y'' + \frac{2x}{(2x+1)(x+1)}y' - \frac{2}{(2x+1)(x+1)}y = \frac{2x+1}{x+1} \tag{2}$$

Let us call the quantity on the right-side of the equation of this equation $g(x)$.

Meanwhile, the Wronskian W for the solutions y_1 and y_2 is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x+1} \\ 1 & -\frac{1}{(x+1)^2} \end{vmatrix} = -\frac{x}{(x+1)^2} - \frac{1}{x+1} = -\frac{2x+1}{(x+1)^2} \tag{3}$$

According to (e.g.) *Differential Equations and Linear Algebra* (2001) by Edwards and Penny, pg. 335 the particular solution to (1) is given by

$$y_p = y_1 u_1 + y_2 u_2 \tag{4}$$

where

$$u_1 = -\int \frac{y_2 g}{W} = -\int \frac{\frac{1}{x+1} \cdot \frac{2x+1}{x+1}}{-\frac{2x+1}{(x+1)^2}} \,\mathrm{d}x = \int \mathrm{d}x = x \tag{5}$$

$$u_2 = \int \frac{y_1 g}{W} = \int \frac{x \cdot \frac{2x+1}{x+1}}{-\frac{2x+1}{(x+1)^2}} \,\mathrm{d}x = -\int x(x+1)\,\mathrm{d}x = -\frac{1}{3}x^3 - \frac{1}{2}x^2 \tag{6}$$

Plugging (5) and (6) into (4), we have

$$\boxed{y_p = x^2 - \frac{1}{x+1}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)} \tag{7}$$

And the overall solution is

$$\boxed{y = c_1 x + \frac{c_2}{x+1} + x^2 - \frac{1}{x+1}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)} \tag{8}$$