Problem #4 (20 points)

Let V be the vector space $P_2[x;R]$ with inner product

$$\langle p,q\rangle = \int_0^1 p(x)q(x)dx$$

and let W be the subspace of all $r(x)$ in V that satisfy the condition

$$\int_0^1 r(x)dx = r(1) - r(0).$$

Decompose the polynomial

$$s(x) = 13 + 26x + 39x^2$$

in V as a sum of one part that lies in entirely in W and one part that is perpendicular to all of W, with respect to the above inner product.

Solution: We begin by noting computing a basis for the subspace W. Since we are working in P_2, we let $r(x) = ax^2 + bx + c$

Applying the boundary value condition, we have

$$\int_0^1 \left(ax^2 + bx + c\right) = r(1) - r(0)$$

$$\left[\frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\right]_0^1 = a + b$$

$$\frac{a}{3} + \frac{b}{2} + c = a + b$$

$$\longrightarrow c = \frac{2}{3}a + \frac{1}{2}b$$

Evidently, $r(x) = ax^2 + bx + \frac{2}{3}a + \frac{1}{2}b$

$$= a\left(x^2 + \frac{2}{3}\right) + b\left(x + \frac{1}{2}\right)$$

This means that the set $\left\{x^2 + \frac{2}{3}, x + \frac{1}{2}\right\}$ constitutes a basis for W. Let us call the vectors in this set w_1 and w_2, respectively. w_1 and w_2 are not orthogonal to each other since $\langle w_1, w_2\rangle \neq 0$. We can convert this into an orthogonal basis by projecting w_1 onto w_2 and redefining w_1 as this projection subtracted from the original vector:

$w_1 \equiv w_1 - \text{proj}_{w_2} w_1$

$$= w_1 - \frac{\langle w_1, w_2 \rangle}{\langle w_2, w_2 \rangle} \cdot w_2$$

$$= x^2 + \frac{2}{3} - \frac{\int_0^1 \left(x^2 + \frac{2}{3}\right)\left(x + \frac{1}{2}\right) \mathrm{d}x}{\int_0^1 \left(x + \frac{1}{2}\right)^2 \mathrm{d}x} \cdot \left(x + \frac{1}{2}\right)$$

$$= x^2 - x + \frac{1}{6}$$

We now compute the components of our vector $s(x)$ parallel to w_1 and w_2 by projecting it onto each of them:

$$s_1 = \mathrm{proj}_{w_1} s$$

$$= \frac{\langle s, w_1 \rangle}{\langle w_1, w_1 \rangle} \cdot w_1$$

$$= -\frac{\int_0^1 \left(13 + 26x + 39x^2\right)\left(x^2 - x + \frac{1}{6}\right) \mathrm{d}x}{\int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 \mathrm{d}x} \cdot \left(x^2 - x + \frac{1}{6}\right)$$

$$= 39x^2 - 39x + \frac{13}{2}$$

$$s_2 = \mathrm{proj}_{w_2} s$$

$$= \frac{\langle s, w_2 \rangle}{\langle w_2, w_2 \rangle} \cdot w_2$$

$$= -\frac{\int_0^1 \left(13 + 26x + 39x^2\right)\left(x + \frac{1}{2}\right) \mathrm{d}x}{\int_0^1 \left(x + \frac{1}{2}\right)^2 \mathrm{d}x} \cdot \left(x + \frac{1}{2}\right)$$

$$= 41x + \frac{41}{2}$$

The sum, $s_{||}$ of the vectors s_1 and s_2

$$\boxed{s_{||} = 39x^2 + 2x + 27}$$

belongs to the subspace W since it is a linear combination of its basis vectors. Meanwhile, the part of s lying outside W is

$$\boxed{s_{\perp} = s - s_{||} = 24x - 14}$$

It is straightforward to confirm that $s_{\perp}$ does not belong to W by showing that it is orthogonal to both w_1 and w_2, i.e. that $\langle s_{\perp}, w_1 \rangle = 0$ and $\langle s_{\perp}, w_2 \rangle = 0$.