CHAPTER 3: Kinematics in Two or Three Dimensions

Vectors

A *vector* is a quantity that has both a *magnitude* and a *direction*. A vector may be represented by an arrow where the length of the arrow represents the magnitude of the vector.

A *scalar* is a quantity that has only a magnitude. To tell vectors and scalars apart, the textbook will always write vector quantities in bold with an over the symbol like this: $\vec{\boldsymbol{A}}$.

If $\vec{\boldsymbol{A}}$ is a vector, then its magnitude can be written simply as $|\vec{\boldsymbol{A}}|$ or simply *A*. The magnitude of a vector is a scalar.

Writing vectors

To write a *specific* vector, one must indicate not only its magnitude but also its direction. For example: “3 blocks in a southwesterly direction” or “40 miles on a bearing of 316°”.

Another way to write vectors is to write them in terms of *components*. For example, suppose that $\vec{\boldsymbol{A}}$ is the vector which stretches from the origin to the point (-2, 4). $\vec{\boldsymbol{A}}$ can be said to have an *x*-component of A_x = -2 and a *y*-component of A_y = 4.

In general, the vectors for most physical quantities will have 3 components—one for each dimension *x*, *y*, and *z*. However, if the vector lies in a plane, the *z*-component is zero and can usually be ignored. In this chapter we will only be mainly concerned with 2-dimensional vectors.

Magnitudes and directions

The magnitude of a vector may be calculated from its components by using the Pythagorean theorem.

$$|\vec{\boldsymbol{A}}| = \sqrt{A_x^2 + A_y^2}$$

If the magnitude and the direction (angle) of a two-dimensional vector are known, as shown in **Figure 1**, the components may be calculated trigonometrically as follows:

$$A_x = A\cos\theta$$
$$A_y = A\sin\theta$$

Conversely, if both components are known, the angle may be calculated using the formula

$$\theta = \mathrm{Tan}^{-1}\left(\frac{A_y}{A_x}\right)$$

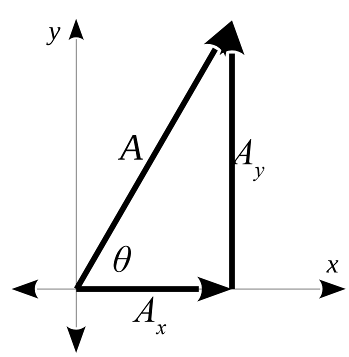

Figure 1: Components of the vector ***A***

Vector sums

To add vectors, just take the tail of one vector and stick it to the head of another as shown in **Figure 2**. The vector sum is the vector that goes from the tail of the first to the head of the last, and is known as the *resultant.*

To calculate the magnitude of the resultant, break each vector into its components and then add each component separately. Thus, if $\vec{\boldsymbol{S}}$ is the sum of the vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$, then

$$S_x = A_x + B_x$$
$$S_y = A_y + B_y$$

Subtracting a vector is equivalent to adding the negative of that vector:

$$\vec{\boldsymbol{A}} - \vec{\boldsymbol{B}} = \vec{\boldsymbol{A}} + (-\vec{\boldsymbol{B}})$$

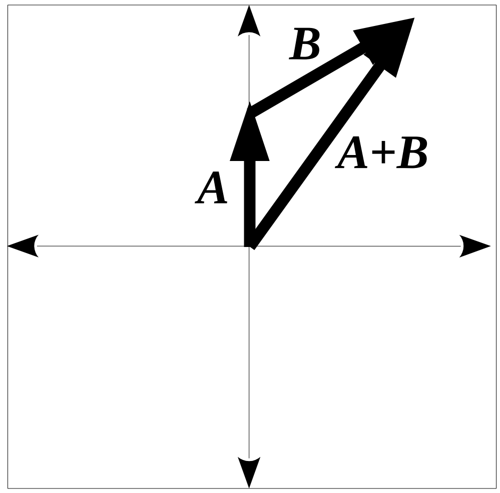

Figure 2: *Vector addition visualized*

where the negative of a vector is the vector with the same length but opposite direction. Thus, if $\vec{\boldsymbol{B}}$ is a vector with a length of 10 meters at a bearing of 230°, it's negative $-\vec{\boldsymbol{B}}$ would a vector with a length of 10 meters pointed at an angle of 50° (50° is 180° from 230°).

Constant acceleration in two dimensions

The laws of kinematics for motion with constant acceleration discussed in the last chapter are equally valid when extended to two or more dimensions. The only difference is that the linear equations of before now become vector equations. Thus, if $\vec{\boldsymbol{r}}$ is a vector,

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_0 + \vec{\boldsymbol{v}}_0 t + \frac{1}{2}\vec{\boldsymbol{a}}\, t^2$$
$$\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_0 + \vec{\boldsymbol{a}}\, t$$

Supposing that the *x*-component of $\vec{\boldsymbol{r}}$ is x and the *y*-component of $\vec{\boldsymbol{r}}$ is *y,* then we can break these equations up into their components:

$$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2 \qquad y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2$$
$$v_x = v_{0x} + a_x t \qquad v_y = v_{0y} + a_y t$$

Since these equations describe motion in just one dimension, the magnitudes may be added like normal numbers (without worrying about the angles).

Projectile motion

A *projectile* is defined as an object that is acted on by gravity and no other forces. Therefore its motion is determined only by its initial conditions (position and velocity) and by the acceleration due to gravity. (Technically, air resistance affects projectile motion too, but we will ignore its effects here, because it makes the math much more complicated. Under conditions of relatively low speed or low atmospheric pressure air resistance is negligible and can be safely neglected anyway.)

The equations of projetile motion are the same as those in the last section, except that the acceleration in the *y* direction is -*g* (*g* stands for 9.8 m/s^2) and the acceleration in the *x*-direction is zero.

$$\boxed{\begin{array}{ll} x = x_0 + v_{0x} t & y = y_0 + v_{0y} t - \frac{1}{2} g t^2 \\ v_x = v_{0x} & v_y = v_{0y} - g t \end{array}}$$

The path taken by a projectile is known as its *trajectory*. The trajectory of a projectile is a section of a parabola. This can be proved by solving the *x* equation for *t* and plugging the resulting expression into the *y* equation (see section 3-7).

It can be shown that the range and maximum height of a projectile (assuming the initial and final heights are the same) are given by

$$R = \frac{v_0^2 \sin(2\theta_0)}{g}$$

$$y_{max} = \frac{v_0^2 \sin^2\theta_0}{2g}$$

Motion in a circle

Uniform circular motion

An object moving in a circular path at constant speed is said to be undergoing *uniform circular motion.* In uniform circular motion, there is no acceleration in the direction of motion at any instant, because any such *tangential* acceleration would cause the speed to increase or decrease. Therefore, the acceleration must be entirely perpendicular to its path. This is referred to as *radial acceleration* or *centripetal acceleration* and is given by

$$a_{rad} = \frac{v^2}{R}$$

where *v* is the speed of the object and *R* is the radius of the circle. The term *centripetal* means "center-seeking" and refers to the fact that the centripetal acceleration always points toward the center of the curve.

The amount of time required for an object to complete one full revolution is referred to as *period* (symoblized T). Since the distance traveled by an object during a revolution is equal to the circumference ($d = 2\pi R$) of its path, the period is related to velocity according to the equation

$$v = \frac{2\pi R}{T}$$

Non-uniform circular motion

If an object is moving in a circular path at *nonconstant* speed, then it has a component of acceleration in both the radial and tangential directions. The *tangential acceleration* is whatever acceleration is in the same direction as the path, namely the rate of change of speed with respect to time.

$$a_{\tan} = \frac{d|\vec{v}|}{dt}$$

Since these accelerations are perpendicular, the overall acceleration *ahagorean* may be calculated using the Pythagorean Theorem:

$$a = \sqrt{a_{\mathrm{rad}}^2 + a_{\tan}^2}$$

Relative velocity

The ground beneath your feet appears to be at rest, but in fact is actually moving at a very high rate of speed due to the rotation of the Earth (about 0.5 km/s at the equator). At the same time, the Earth is moving around the sun at an even higher speed. Relative to a stationary object on the Earth's surface, you are at rest, but relative to an astronaut positioned direclty above the sun, you would be seen to be moving at nearly 30 km/s.

All velocities are *relative*. In other words, there is no one "correct" way of measuring velocity; instead one must be clear about *what the velocity is being measured relative to*. To indicate this, physicists often use subscripts. For example, the symbol $\vec{v}_{\mathrm{PT}}$ could indicate the velocity of a *p*erson moving in a the aisle of a moving *t*rain while the symbol $\vec{v}_{PG}$ could represent the velocity of that same person relative to the *g*round below the train.

If the person is walking at 5 km/h to the west and the train is moving at 60 km/h to the east, their overall velocity relative to the ground would be equal to the vector sum of $\vec{v}_{\mathrm{PT}}$ and $\vec{v}_{\mathrm{TG}}$ which is 55 km/h to the east.

More generally, the velocity of an object A relative to an object B is given by

$$\boxed{\vec{v}_{\mathrm{AB}} = \vec{v}_{\mathrm{AC}} + \vec{v}_{\mathrm{CB}}}$$

where C is an object whose velocity can be measured relative to both A and B. (Note that for this equation to work, the inner subscripts of the added velocities must be the same, and the outer subscripts of the two added velocities must be the same as the subscripts of the velocity you desire to calculate.)