Momentum and impulse

The momentum of a particle at a given time is defined as the product of its mass with its velocity and is measured in Newton-seconds (N·s).

$$\boxed{\vec{p} = m\vec{v}}$$

Impulse is defined as the definite integral of force with respect to time and is equal to the change in momentum

$$\vec{J} = \int_{t_1}^{t_2} \vec{F}\,dt = \int_{t_1}^{t_2} m\frac{d\vec{v}}{dt} = \int_{t_1}^{t_2} m\,d\vec{v} = \Delta\vec{p}$$

Although the exact equations that describe the forces during a collision may be complicated, depending on the details such as the shape and material properties of the objects involved, if all we care about is the *average* force, the above equation may be summarized by

$$\boxed{F\Delta t = m\Delta v}$$

More generally, by equating the second and last terms in the above derivation, we see that momentum is the definite integral of the force: $\int \vec{F}\,dt = \Delta\vec{p}$, and therefore,

$$\boxed{\vec{F} = \frac{d\vec{p}}{dt}}$$

Conservation of momentum

Just as the energy of a system is constant as long as no outside work is done on or by it, *the total momentum of a system is constant as long as the vector sum of external forces is zero.* Assuming there are no net external forces, then $J = 0$, which means that $\Delta\vec{p} = 0$. The conservation of momentum is in a sense more general than that of energy, because there is only one kind of momentum, whereas energy can be converted into multiple different forms. Moreover, unlike the conservation of mechanical energy, the conservation of momentum applies even when some of the forces acting on an object are nonconservative.

Just as a dropped object *appears* to lose energy when it hits the floor and stops, there are some situations which make it *appear* that momentum is not conserved. For example, when a pendulum swings back and forth, its momentum changes direction repeatedly from one direction to another. However, if the entire momentum of the earth-pendulum system as a whole is considered, then its momentum is indeed constant. It would be inconvenient to use conservation of momentum to analyze the momentum of the pendulum in this situation, however, because the effect of the pendulum on the Earth's velocity is not easily measured.

Because of situations like these it is important to take a moment to consider when starting a problem whether it is more appropriate to apply conservation of energy or conservation of momentum.

Collisions

Conservation of momentum is particularly useful when analyzing collisions, because if the collision is of short duration, external forces are likely to be minimal and thus it can be assumed that momentum is conserved. In the case of two cars of equal mass hitting each other, some of their initial kinetic energy is converted into heat and/or the breaking of chemical bonds during the deformation of the materials. If this deformation is not permanent, some or all of this energy may be reinjected into the system, giving the objects a final velocity relative to each other.

According to the law of conservation of momentum, if two objects with masses m_A and m_B and velocities $\vec{v}_A$ and $\vec{v}_B$ collide, their final velocities $\vec{v}'_A$ and $\vec{v}'_B$ must be related by the following equation

$$\boxed{m_A \vec{v}_A + m_A \vec{v}_A = m_A \vec{v}'_A + m_B \vec{v}'_B}$$

The initial conditions of the collision do not contain enough information to predict the velocities of the objects after the collision, unless one of the final velocities is known. However, the final velocities can be calculated if the collision's *coefficient of restitution* (COR) is known. The COR is defined as the ratio of the relative speed after to the collisions to the relative speed before the collision:

$$\boxed{C_R = -\frac{v'_B - v'_A}{v_B - v_A}}$$

A collision may be classified according to the COR as follows:

- $C_R = 1$: *elastic*
- $0 < C_R < 1$: *inelastic*
- $C_R = 0$: *completely inelastic*

In an elastic collision, the relative speed and kinetic energy are conserved. In completely inelastic collisions, the final relative velocity of the objects is zero (i.e. they stick together). In all other collisions the relative speed of the objects after the collision will be less than or equal to the initial relative speed, and therefore, the COR can never be greater than 1.

In collisions involving motion in more than one dimension, the conservation of momentum equation must be applied once for each dimension in which there is motion.

Center of mass

Up to now, we have been considering all objects to be *point particles* (i.e. objects whose only characteristics are their mass and their location). However, real objects not have not only masses and locations, but also three-dimensional shapes and orientations, and may rotate, stretch, or even break apart or combine with other objects.

However, even for real objects, there is one point in the object whose motion is exactly the same as that of a point particle whose mass is equal to the total mass of all the parts of the object. That point is known as the *center of mass*. When a hammer is tossed in the air, its center of mass will move in a parabolic path, as would a point particle, even if the other particles in the hammer move irregularly.

If all forces on an object were to act directly through the center of mass, then the motion of the center of mass and the overall object would be the same. This is the reason that in previous chapters we were able to apply Newton's second law to objects weren't particles; the effect of the net force is the same either way.

A *rigid* body is an object where the positions of the constituent particles are fixed in space relative to each other. A rigid body may move translationally or rotate, but the distance between its particles and its center of mass does not change. But even for non-rigid objects, the motion of the center of mass is particle-like, as long as there are no external forces on the object. Thus, if a projectile were to explode in mid-flight, causing it to fragment into multiple pieces, the center of mass of its scattered remains after landing would be the same as that of the projectile had it remained intact.

Center of mass defined precisely as the *weighted average* of position with respect to mass. Suppose that we have a collection of particles (x_i, y_i) each of which has a mass m_i,. Then the *x*- and *y*-components of the center of mass are given as follows:

$$x_{CM} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \qquad y_{CM} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$$

where the denominator is just the total mass of the entire system. Since the numerator of this equation is a sum of masses multiplied by positions and the denominator is a mass, center of mass has units of position.

The above statement is closely related to the conservation of momentum. If no net force is acting on a system of two or more masses, then its center of mass will be moving with a constant velocity. Hence, if we differentiate the above equation with respect to time, and multiply both sides by the total mass, we will see that the sum of the momenta is indeed equal to a constant.

For a continuously defined mass distribution, the above equation can be generalized to

$$x_{CM} = \frac{\int_{x_1}^{x_2} \rho A\, x\, dx}{\int_{x_1}^{x_2} \rho A\, dx}\ , \quad y_{CM} = \frac{\int_{y_1}^{y_2} \rho A\, y\, dy}{\int_{y_1}^{y_2} \rho A\, dy}\ , \text{ etc.}$$

Where $\rho(x, y, z)$ is the density function and A is the cross-sectional area function perpendicular to the axis of integration.