Chapter 9: Rotation of rigid bodies

A *rigid* body has a definite shape and size. It may move or rotate but it will not stretch, twist, or be deformed in any way.

Assuming there are no external forces, all motion of a rigid body may be divided into two parts: motion *of* the center of mass (translation) and motion *around* the center of mass (rotation).

$$x(t) = x_{\text{cm}}(t) + x_{\text{rot}}(t)$$

In other words, any desired orientation of a rigid object may be produced by first translating it and then rotating it.

We'll deal with how forces affect rotation (rotational dynamics) in Chapter 10. This chapter largely ignores the *causes* of motion and studies the motion itself.

Angular coordinates

Unlike translational motion, rotational motion cannot be modeled adequately by a particle. We need to consider the object's *internal* motion as well as its *external* motion.

Consider a disk rotating around a fixed axis. Any point on the disk may be represented by two coordinates: a radius (r) and an angle (θ). By convention, θ starts at zero on the x-axis and increases in a counterclockwise direction.

We could have used x and y instead, but r and θ are much more convenient for representing rotational motion, because for a particular point on the object, x and y are continuously changing at varying rates, while r is constant and θ is changing at a constant rate. You can interconvert Cartesian (x and y) coordinates and polar coordinates with the following formulas:

$$\begin{array}{ll} r = \sqrt{x^2 + y^2} & x = r\cos\theta \\ \theta = \arctan\dfrac{y}{x} & y = r\sin\theta \end{array}$$

The actual distance traveled by a point on a rotating object is called its *arc length* (s). Arc length is proportional to both the radius and the angle. Angles can be measured in degrees, revolutions, or in radians, but the most convenient unit is the radian because then the constant of proportionality is 1.

$$s = r\theta$$

One radian is defined as the angle for which the radius and arc length are equal to each other (or about 57.3°—a bit less than 1/6 of a revolution).

One full revolution equals 360° or 2π rad, so to convert from degrees to radians, multiply by

$$\frac{2\pi}{360} = \frac{\pi}{180}$$

Angular velocity

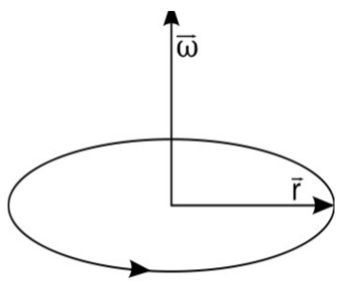
Instantaneous angular velocity is defined as the rate of change of angle with respect to time.

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Similarly, average angular velocity is defined by

$$\omega_{\mathrm{avg}} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

If ω is represented as a vector quantity, its direction is given by the right hand rule, so that the fingers curl in the direction of rotation and the thumb points in the direction of $\vec{\omega}$



The tangential velocity of a point on a rotating disk may be related to the angular velocity by differentiating $s = r\theta$:

$$v_{\tan} = \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}r\theta}{\mathrm{d}t} = r\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{\mathrm{d}r}{\mathrm{d}t}\theta$$

Assuming that the distance from the to the center is constant (i.e. that the object is rigid), the second term vanishes and we have

$$v_{\tan} = \omega r$$

Differentiating again (still assuming rigidity) we get

$$a_{\tan} = \alpha r$$

where

$$\alpha \equiv \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$$

is the *angular acceleration*

Meanwhile, the radial (centripetal) acceleration may be rewritten in terms of ω:

$$a_{\mathrm{rad}} = \frac{v^2}{r} = \omega^2 r$$

The kinematic equations for angular motion under constant angular acceleration are analogous to the equations for linear motion with *x* replaced by *θ*, *v* by *ω*, and *a* by *α*.

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\omega = \omega_0 + \alpha t$$
$$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

Rotational energy

In a non-rotating rigid object, each mass element has the same velocity and therefore, the total kinetic energy is just one half of the total mass multiplied by the square of the velocity.

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2}\left(\sum_i m_i\right) v^2 = \frac{1}{2} M v^2$$

For a rotating object, however, the velocity varies with respect to distance from the axis, so the total kinetic energy will depend on the geometry of the object.

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2$$

The quantity in the parentheses is known as the *moment of inertia* and is represented by the symbol *I*.

$$I \equiv \sum_i m_i r_i^2$$

In the case of a continuously defined mass, this becomes

$$I = \int r^2 \, \mathrm{d}m$$

In terms of I, the kinetic energy is given by

$$K = \frac{1}{2} I \omega^2$$

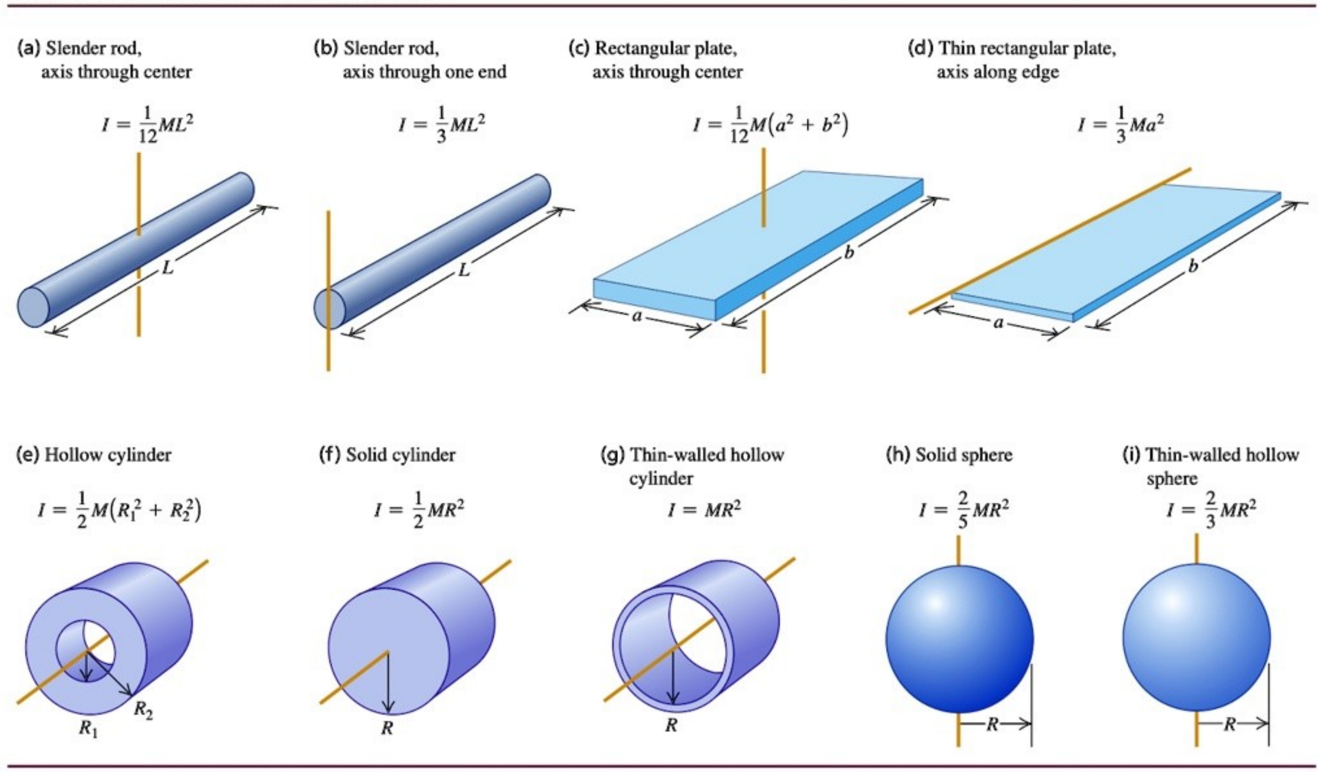
The moment of inertia is so-called because it represents the difficulty of changing an objects angular velocity, just as ordinary inertia (mass) represents the difficulty of changing an objects linear velocity.

For a cylinder (or ring) whose thickness is much less than its radius, we can assume that all its mass is located at a distance *R* from the axis of rotation, and therefore, its moment of inertia is simply MR^2.

On the other hand, we can show (using calculus)that the moment of inertia of a solid cylinder or disk is $\frac{1}{2}MR^2$.

Table 9.2 in *Sears and Zemansky*, reproduced below, gives the moment of inertia of several commonly encountered shapes in physics. In general, the further the mass is from the axis of rotation, the greater the moment of inertia.

TABLE 9.2 Moments of Inertia of Various Bodies



Parallel axis theorem

Notice that the moment of inertia of an object depends on the location of the axis. For example, if you rotate a rod of length L about one of its endpoints, its moment of inertia is $\frac{1}{3}ML^2$ but if you rotate it around its midpoint, its moment of inertia is only $\frac{1}{12}ML^2$.

If the moment of inertia through an objects center of mass is known, then the moment of inertia about some other axis parallel to the first can be calculated as follows

$$I_{\mathrm{P}} = I_{\mathrm{CM}} + Md^2$$

Where *d* is the distance between the axis as the center of mass. This useful equation is known as the parallel axis theorem.