A train with length L_0, as measured in the train's rest frame S', is moving toward a tunnel with velocity $\beta = v/c$ in the $+x$-direction. The tunnel has length L_0 in the rest frame of the Earth, S.

(a) For an observer standing on the side of the tracks, what is the observed length of the train? From the point of view of this observer, will there be a point in time in which the train has completely disappeared into the tunnel and no part of it protrudes?

(b) For an observer within the train, what is the observed length of the tunnel? From the point of view of this observer, will there be a point at which the train has completely disappeared into the tunnel?

The answers you've obtained should present a paradoxical situation. Suppose that within the reference frame of an observer at the side of the tracks, two doors simultaneously open and close at the entrance and exit of the tunnel immediately after the back of the train enters the tunnel.

(c) Suppose that the entrance door closes at $t = t' = 0$ and $x = x' = 0$. At what time and position does the exit door close in the S frame?

(d) At what time and position does the exit tunnel door close in the S' frame?

(e) How does one resolve this paradox?

(a) In the tunnel's reference frame, the length of the train is less than that of the tunnel because $L < L_0$ for all $\beta > 0$.

$$L_{\text{train}} = \frac{L_0}{\gamma} = L_0\sqrt{1-\beta^2}$$

Were γ is the standard Lorentz factor

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

(b) In the train's reference frame the length of the tunnel is less than that of the train by the same factor and for the same reason.

$$L_{\text{tunnel}} = \frac{L_0}{\gamma}$$

(c) Since the tunnel is at rest in the S frame, no transformations are required.

$$x = L_0$$

$$t = 0$$

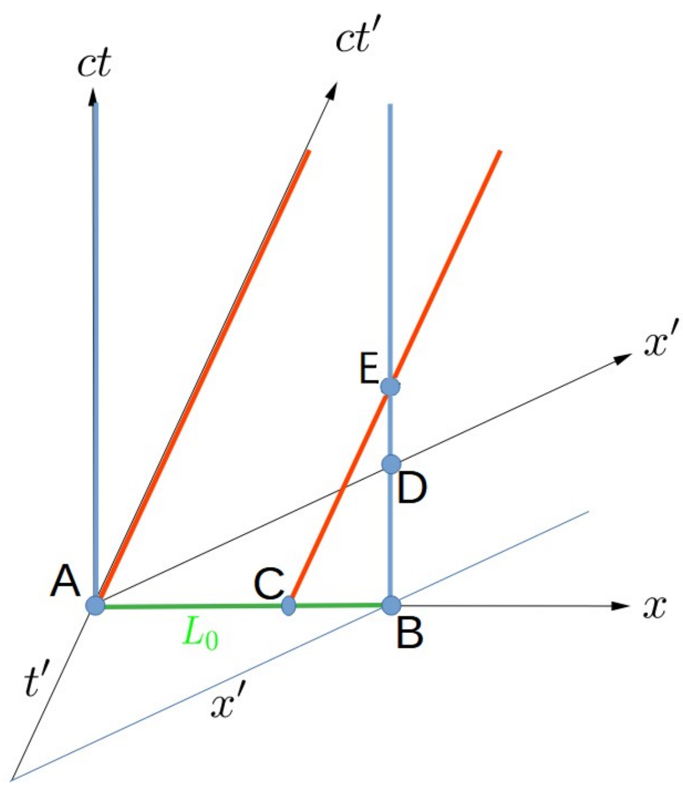


Figure 1: Spacetime diagram showing the motion of the train and tunnel. The blue lines and red lines show the positions of the tunnel and train, respectively. The events A and B represent the closing of the entrance and exit doors, respectively.

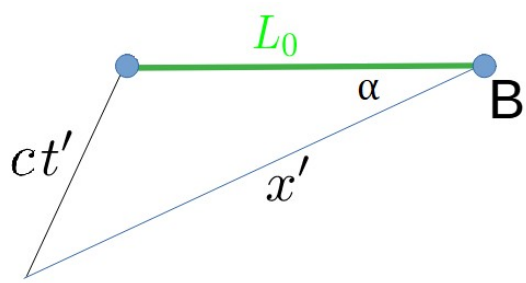
(d) In S' the spacetime coordinates are as follows:

$$x' = \gamma L_0 = \frac{L_0}{\sqrt{1-\beta^2}}$$

$$t' = -\frac{\gamma v L_0}{c^2} = -\frac{\gamma \beta L_0}{c} = -\frac{L_0}{c\sqrt{\beta^{-2}-1}}$$

This can be shown easily using a Lorentz transform. A more intuitive approach is to use the spacetime diagram shown in Figure 1.

Let us focus our attention on the triangle below the x-axis which has been constructed in order to compute the coordinates in the tilted coordinate system.



Where $\alpha = \arctan \beta$ is the angle between the coordinate systems. From this the unknown side lengths x' and t' can be computed straightforwardly using the Law of Sines along with other common trig formulas.

$$\frac{x'}{\sin\left(\frac{\pi}{2}+\alpha\right)} = \frac{L_0}{\sin\left(\frac{\pi}{2}-2\alpha\right)} \longrightarrow x' = \frac{L_0\sqrt{\beta^2+1}}{1-\beta^2}$$

$$\frac{t'}{\sin\alpha} = \frac{L_0}{\sin\left(\frac{\pi}{2}-2\alpha\right)} \longrightarrow t' = \frac{\beta L_0}{c}\frac{\sqrt{\beta^2+1}}{\beta^2-1}$$

It is important to note however that these lengths are measured in the coordinate system of S in which L_0 is the length of the tunnel. In order to get the lengths in the S' system, the results must be scaled by

$$\sqrt{\frac{1-\beta^2}{1+\beta^2}}$$

which in Figure 1 is the ratio between $\overline{AC}$ and $\overline{AD}$, each of which represents the length of the train.

Multiplying by this scale factor gives the results previously stated.

(e) In the frame of the tunnel the front of the train does not reach the exit door (event E in the figure) door until a time $t = \frac{L_0 - L}{v}$ after the doors close because the train has to travel an additional distance $L_0 - L$.

The apparent paradox is that if in the train's reference frame the tunnel is shorter than the train, then the door B will close on the train while the train is still passing through the exit. This would contradict the previous paragraph since the door cannot both hit the train and not hit the train.

The solution to this "paradox" is to note that in the frame of the train the doors do not close simultaneously. Door B closes *before* door A. This means that even though the tunnel is shorter than train, the train still has to some distance to travel before it will collide with the door.

There is no contradiction; both perspectives are shown in the spacetime diagram.