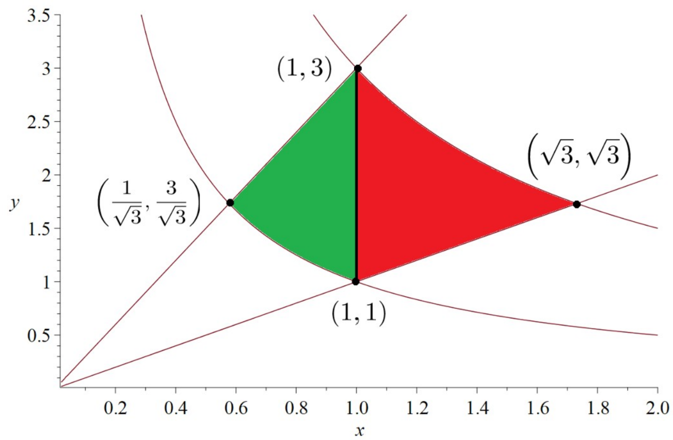
Q: Find $\displaystyle\int\int_R xy\,\mathrm{d}A$ where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$ and $y = 3$.

Method 1 (hard): Cartesian coordinates



We can divide the region into two subregions separated by the vertical line $x = 1$ and integrate each separately.

The green region is bounded above by $y = 3x$ and below by $y = \frac{1}{x}$. The red region is bounded above by $\frac{3}{x}$ and below by $y = x$. Using the area element $\mathrm{d}A = \mathrm{d}y\,\mathrm{d}x$ we have.

$$\int_{\frac{1}{\sqrt{3}}}^{1}\int_{\frac{1}{x}}^{3x} xy\,\mathrm{d}y\,\mathrm{d}x + \int_{1}^{\sqrt{3}}\int_{x}^{\frac{3}{x}} xy\,\mathrm{d}y\,\mathrm{d}x$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} x\,\mathrm{d}x\int_{\frac{1}{x}}^{3x} y\,\mathrm{d}y + \int_{1}^{\sqrt{3}} x\,\mathrm{d}x\int_{x}^{\frac{3}{x}} y\,\mathrm{d}y$$

$$= \frac{1}{2}\int_{\frac{1}{\sqrt{3}}}^{1} x\left[9x^2 - \frac{1}{x^2}\right]\mathrm{d}x + \frac{1}{2}\int_{1}^{\sqrt{3}} x\left[\frac{9}{x^2} - x^2\right]\mathrm{d}x$$

$$= \frac{1}{2}\left[\frac{9x^4}{4} - \ln(x)\right]_{\frac{1}{\sqrt{3}}}^{1} + \frac{1}{2}\left[9\ln(x) - \frac{x^4}{4}\right]_{1}^{\sqrt{3}}$$

$$= \left(\frac{9}{8} - \frac{1}{8} + \frac{1}{2}\ln\frac{1}{\sqrt{3}}\right) + \left(\frac{9}{2}\ln\sqrt{3} - \frac{9}{8} + \frac{1}{8}\right)$$

$$= -\frac{1}{4}\ln 3 + \frac{9}{4}\ln 3$$

$$= \boxed{2\ln 3}$$

Method 2 (harder): Radial coordinates

Using radial coordinates we only need one integral. The integrand xy becomes $r^2 \cos\theta \sin\theta = \frac{1}{2} r^2 \sin(2\theta)$. Meanwhile the angle lies between $\theta = \frac{\pi}{4}$ and $\arctan 3$, and the curves $xy = 1$ and $xy = 3$ correspond to $r = \sqrt{2\csc(2\theta)}$ and $r = \sqrt{6\csc(2\theta)}$ respectively. Using the area element $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$, our integral becomes

$$\int_{\frac{\pi}{4}}^{\arctan 3} \int_{\sqrt{2\csc(2\theta)}}^{\sqrt{6\csc(2\theta)}} \frac{1}{2} r^3 \sin(2\theta)\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\arctan 3} \sin(2\theta)\,\mathrm{d}\theta \int_{\sqrt{2\csc(2\theta)}}^{\sqrt{6\csc(2\theta)}} r^3\,\mathrm{d}r$$

$$= \frac{1}{8}\int_{\frac{\pi}{4}}^{\arctan 3} \sin(2\theta)\big(36\csc^2(2\theta) - 4\csc^2(2\theta)\big)\,\mathrm{d}\theta$$

$$= -4\int_{\frac{\pi}{4}}^{\arctan 3} \csc(2\theta)\,\mathrm{d}\theta$$

$$= 2\left[\ln(\csc(2x) - \cot(2x))\right]_{\frac{\pi}{4}}^{\arctan 3}$$

$$= 2\ln(\csc(2\arctan(3))) - 2\ln(\cot(2\arctan(3)))$$

$$= 2\ln\left[\frac{1}{2\sin(\arctan 3)\cos(\arctan 3)} - \frac{\cot^2(\arctan 3) - 1}{2\cot(\arctan 3)}\right]$$

$$= 2\ln\left[\frac{1}{2\cdot\frac{3}{\sqrt{10}}\frac{1}{\sqrt{10}}} - \frac{\frac{1}{9} - 1}{2\cdot\frac{1}{3}}\right]$$

$$= 2\ln\left(\frac{5}{3} + \frac{4}{3}\right)$$

$$= \boxed{2\ln 3}$$

Method 3 (easy):

Through inspired choice, make the coordinate transformation $x = \frac{u}{v}$ and $y = v$.
The boundaries of the region become

$$
\begin{array}{lcl}
xy = 1 & \longrightarrow & u = 1 \\
xy = 3 & \longrightarrow & u = 3 \\
y = x & \longrightarrow & v = \sqrt{u} \\
y = 3x & \longrightarrow & v = \sqrt{3u}
\end{array}
$$

The integrand xy becomes simply u and the area element is

$$\mathrm{d}A = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathrm{d}u \, \mathrm{d}v = \begin{vmatrix} \frac{1}{v} & 1 \\ 0 & 1 \end{vmatrix} \mathrm{d}u \, \mathrm{d}v = \frac{1}{v} \, \mathrm{d}u \, \mathrm{d}v$$

Putting it all together, we have

$$\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v} \, \mathrm{d}v \, \mathrm{d}u$$

$$= \int_1^3 u \int_{\sqrt{u}}^{\sqrt{3u}} \frac{\mathrm{d}v}{v}.$$

Since the integrals are independent we can calculate them separately and get

$$= \boxed{2 \ln 3}$$