4. Find the power series for $f(x) = \frac{1}{1+2x}$ centered at $x = 0$. What is the radius of convergence?

By anaology with the summation formula for an infinite geometric series from pre-calculus,

$1 + r + r^2 + r^3 + \cdots = \dfrac{1}{1-r}$ where $|r| < 1$

we can define $2x = -r$ so that r = -2x

Substituting this into the summation formula we have

$$1 + (-2x) + (-2x)^2 + (-2x)^3 + \cdots = \frac{1}{1+2x}$$

so that

$$\frac{1}{1+2x} = 1 - 2x + 4x^2 - 8x^3 \cdots$$

or

$$\frac{1}{1+2x} = \boxed{\sum_{n=0}^{\infty} (-2)^n x^n}$$

Radius of convergece: Since the geometric series converges when $|r| < 1$, this implies that the power series will converge when $|-2x| < 1$, i.e.

$$\boxed{|x| < \frac{1}{2}}$$

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5. Find the power series for xe^{3x} centered at $x = 0$. What is the radius of convergence?

We start with the well-known Taylor series expansion for e^x:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

Replacing x with 3x and substituting into our function, we have

$$xe^{3x} = x\left(1 + 3x + \frac{(3x)^2}{2} + \frac{(3x)^3}{6} + \cdots\right) = x\sum_{n=0}^{\infty} \frac{3^n x^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{3^n x^{n+1}}{n!}}$$

Radius of convergence

The ratio of subsequent terms in the expansion is $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{\frac{3^{n+1}x^{n+2}}{n+1!}}{\frac{3^n x^{n+1}}{n!}}\right| = \left|\dfrac{3x}{n+1}\right|$

Since the ratio approaches 0 as $n \to \infty$, the series converges everywhere. The radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.

6a. Find the power series for $\cos x - 1 + \frac{x^2}{2}$ centered at $x = 0$. What is the radius of convergence?
b. Use part a to estimate

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} \ .$$

a) We can can use the Taylor series expansion for $\cos x$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

so that

$$\cos x - 1 + \frac{x^2}{2} = \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots = \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Radius of convergence:

The ratio of subsequent terms in the expansion is $\left|\frac{a_{n+1}}{a_n}\right|$

$$= \frac{\frac{x^{2(n+1)}}{(2(n+1))!}}{\frac{x^{2n}}{(2n)!}} = \frac{x^2}{4(n+1)(n)}$$

Again, as a n becomes large, the ratio approaches zero, so this converges everywhere and $R = \infty$.

b) Substituting the power series expansion into the limit gives

$$\lim_{x\to 0} \frac{\frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots}{x^4}$$

As $x \to 0$, the $n = 6$ and greater terms will fall off more rapidly than x^4, so the limit appraoches

$$\lim_{x\to 0} \frac{\frac{x^4}{4!}}{x^4} = \lim_{x\to 0} \frac{1}{4!} = \boxed{\frac{1}{24}}$$

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7. For what interval of $x-$values does this series converge?

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{3^n}$$

This is a power series with ratio $r = \frac{x-4}{3}$, so it converges whenever $|r| < 1$

This means $\left|\frac{x-4}{3}\right| < 1 \longrightarrow |x-4| < 3 \longrightarrow 1 < x < 7$, so the interval of convergence is $\boxed{(1,7)}$