

Question: Use an *integrating factor* to solve the following first-order differential equation

$$x^3 y' - x^2 y = 3x \quad (1)$$

Solution: The goal is to manipulate the left side of (1) to become the derivative of a product. First, let's simplify by dividing both sides by x^3 .

$$y' - \frac{1}{x}y = \frac{3}{x^2} \quad (2)$$

Next we multiply both sides by the (unknown) integrating factor $\mu(x)$

$$\mu y' + -\frac{\mu}{x}y = \frac{3}{x^2}\mu \quad (3)$$

By definition, $\mu(x)$ has the property that the left side of (3) will become

$$\frac{d}{dx}(\mu y) = \mu y' + \mu' y \quad (4)$$

Comparing the RHS of (4) with the LHS of (3) with the last, we see that

$$\mu' = -\frac{\mu}{x} \quad (5)$$

This is a separable differential equation that can be solved as follows:

$$\frac{d\mu}{\mu} = -\frac{dx}{x}$$

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$$\ln \mu = -\ln x$$

$$\mu = e^{-\ln x}$$

$$\mu = (e^{\ln x})^{-1}$$

$$\mu = x^{-1} \quad (6)$$

(Note that it is okay to ignore the constant of integration (+C) since we only need a solution—not the general solution.) Now that we have solved for $\mu(x)$, we can, according to equation (4), replace the LHS of (3) with $\frac{d}{dx}(\mu y)$. This results in

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = \frac{3}{x^3} \quad (7)$$

This can be easily solved by integrating both sides:

$$\begin{aligned}y &= x \int \frac{3}{x^3} dx \\&= 3x \left(-\frac{1}{2}x^{-2} + C \right) \\&\boxed{y = -\frac{3}{2x} + Cx} \quad (8)\end{aligned}$$

Final note: In the future, to save time solving for $\mu(x)$, we can memorize that that if the differential equation is written in the form

$$y' + p(x)y = q(x) \quad (9)$$

then the integrating factor can be written

$$\boxed{\mu(x) = e^{\int p(x)dx}} \quad (10)$$

and the general solution will be

$$\boxed{y = e^{-\int p(x)dx} \int q(x)e^{\int p(x)dx} dx} \quad (11)$$