Q: Solve the differential equation

$$y' = xy \tag{1}$$

using the power series method.

A: We assume a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n \tag{2}$$

with corresponding derivatives

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \tag{3}$$

Substituting (2) and (3) into (1) gives

$$\sum_{n=1}^{\infty} n c_n x^{n-1} = x \sum_{n=0}^{\infty} c_n x^n \tag{4}$$

We can simplify this by distributing the x's on the right-hand side into the sum:

$$\sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} c_n x^{n+1} \tag{5}$$

In order to get a relation between various c_n, we need to match the powers in each series. Let us define a new index m such that

$$n = m + 2 \tag{6}$$

Rewriting the left-hand side of (5) in terms of m gives

$$\sum_{m=-1}^{\infty} (m+2) c_{m+2} x^{m+1} \tag{7}$$

Since m is an arbitrary label, so we might as well call it n as before. As a result, (5) becomes

$$\sum_{n=-1}^{\infty} (n+2) c_{n+2} x^{n+1} = \sum_{n=0}^{\infty} c_n x^{n+1} \tag{8}$$

The first term on the left-hand side, corresponding to $n = -1$, is just c_1. We can separate this from the overall sum so that the indices of both summations match:

$$c_1 + \sum_{n=0}^{\infty}(n+2)c_{n+2}x^{n+1} = \sum_{n=0}^{\infty} c_n x^{n+1} \tag{9}$$

Since there are no constant terms on the right-hand side, the only way that (9) can be true for all values of x is if $c_1 = 0$. We can make a table by equating the other powers of x on the left and right sides of (9) as for various values of n:

n	power	left	right	recurrence relation
0	x^1	$2c_2 x$	$c_0 x$	$c_2 = \dfrac{1}{2}c_0$
1	x^2	$3c_3 x^2$	$c_1 x^2$	$c_3 = \dfrac{1}{3}c_1$
2	x^3	$4c_4 x^3$	$c_2 x^3$	$c_4 = \dfrac{1}{4}c_2$
3	x^4	$5c_5 x^4$	$c_3 x^4$	$c_5 = \dfrac{1}{5}c_3$
n	x^{n+1}	$(n+2)c_{n+2}x^{n+1}$	$c_n x^{n+1}$	$c_{n+2} = \dfrac{1}{n+2}c_n$

Since $c_1 = 0$, all of the odd terms in the summation vanish. Meanwhile there are no restrictions on c_0, so it is an arbitrary constant. The even terms can be written in terms of c_0 as follows:

$$c_2 = \frac{1}{2}c_0, \qquad c_4 = \frac{1}{4}c_2 = \frac{1}{4\cdot 2}c_0, \qquad c_6 = \frac{1}{6}c_4 = \frac{1}{6\cdot 4\cdot 2}c_0, \qquad c_8 = \frac{1}{8}c_6 = \frac{1}{8\cdot 6\cdot 4\cdot 2}c_0$$

or in general

$$c_n = \frac{A}{n!!} \tag{10}$$

where A is arbitrary and

$$n!! = n \cdot (n-2) \cdot (n-4) \cdots \tag{11}$$

is the double factorial function. Substituting (10) back into the solution (2) for even values of n gives our final answer:

$$\boxed{y = A \sum_{n \text{ even}}^{\infty} \frac{x^n}{n!!}} \tag{12}$$

Note: The differential equation (1) can also be solved by other methods. For example, we can apply separation of variables to (1) to get

$$\frac{\mathrm{d}y}{y} = x\,\mathrm{d}x \tag{13}$$

Integrating both sides yields

$$\ln|y| = \frac{1}{2}x^2 + C$$

or

$$y = Ae^{x^2/2} \tag{14}$$

Is this equivalent to our earlier solution (12)? We can see that it is because if we replace the exponential in (14) with the Taylor series

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \tag{15}$$

and substitute $x \longrightarrow \frac{x^2}{2}$, the result is identical to what we got before.