Q: Find the wavefunction of a particle in a 1-dimensional delta function potential

$$V(x) = -\alpha\delta(x). \tag{1}$$

A: Plugging (1) into the time-independent Schrödinger equation

$$\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + V(x)\psi(x) = E\psi(x) \tag{2}$$

results in the eigenvalue equation

$$\left\{\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha\delta(x) - E\right\}\psi(x) = 0 \tag{3}$$

While the delta function doesn't have a well-defined value at $x = 0$, its integral does. Therefore we integrate (3) from $-\epsilon$ to ϵ, where ϵ is a very small number. This results in

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}\,\mathrm{d}x - \alpha\int_{-\epsilon}^{\epsilon}\psi(x)\delta(x)\,\mathrm{d}x - E\int_{-\epsilon}^{\epsilon}\psi(x)\,\mathrm{d}x \tag{4}$$

The first integral is simply the derivative of ψ evaluated immediately on either side of the potential while he second integral becomes $\alpha\psi(0)$ due to the sifting property of the delta function. The third integral vanishes since ϵ is small and $\psi(x)$ must be non-infinite. Altogether (4) becomes

$$-\frac{\hbar^2}{2m}\left(\left.\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|_{\epsilon} - \left.\frac{\mathrm{d}\psi}{\mathrm{d}x}\right|_{-\epsilon}\right) - \alpha\psi(0) = 0 \tag{5}$$

Since the potential is symmetric with respect to the origin, we guess a solution of the form

$$\psi(x) = Ae^{-\kappa|x|} \tag{6}$$

Substituting our trial wavefunction (6) into (5) results in a valid solution provided that

$$-\frac{\hbar^2}{2m}(-2\kappa A) - \alpha A = 0$$

or

$$\kappa = \frac{\alpha m}{\hbar^2} \tag{7}$$

Finally we apply the normalization condition to (6) which gives

$$A = \sqrt{\kappa} = \sqrt{\frac{\alpha m}{\hbar^2}}$$

so that the final wavefunction takes the form

$$\boxed{\psi(x) = \sqrt{\frac{\alpha m}{\hbar^2}}e^{-\alpha m|x|/\hbar^2}} \tag{8}$$

Evidently there is a single bound state. Its energy can be found in the usual way

$$
\begin{aligned}
\langle E \rangle &= \langle \psi | H | \psi \rangle \\
&= \int_{-\infty}^{\infty} \psi \left(\frac{\hbar^2}{2m} \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha\delta(x) \right) \psi \,\mathrm{d}x \\
&= \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \psi^2 \kappa^2 \,\mathrm{d}x - \alpha \int_{-\infty}^{\infty} \psi^2 \delta(x) \,\mathrm{d}x \\
&= 2 \cdot \frac{\hbar^2 \kappa^2}{2m} \int_0^{\infty} \kappa e^{-2\kappa x} \,\mathrm{d}x - \alpha \psi^2(0) \\
&= \frac{\hbar^2 \kappa^3}{2m\kappa} \left[e^{-2\kappa x} \right]_0^{\infty} - \alpha\kappa \\
&= \frac{\hbar^2 \kappa^2}{2m} - \alpha\kappa \\
&= \frac{\alpha^2 m}{2\hbar^2} - \frac{\alpha^2 m}{\hbar^2} \\
&= \boxed{-\frac{\alpha^2 m}{2\hbar^2}}
\end{aligned}
$$