Q: The Hermitian operator $\hat{\Omega}$ corresponding to the observable Ω has three eigenstates $|\omega_1\rangle$, $|\omega_2\rangle$ and $|\omega_3\rangle$ with distinct eigenvalues ω_1, ω_2 and ω_3, respectively. Show that $\hat{\Omega}$ can be written in the form

$$\hat{\Omega} = \omega_1 |\omega_1\rangle\langle\omega_1| + \omega_2 |\omega_2\rangle\langle\omega_2| + \omega_3 |\omega_3\rangle\langle\omega_3| .$$

A: Let $\hat{\Omega}$ operate on the arbitrary state $|\psi\rangle$.

$$\hat{\Omega} |\psi\rangle$$

Since $|\psi\rangle$ has three distinct eigenvectors, we will assume that they comprise an orthonormal eigenbasis, $\psi = c_1 |\psi_1\rangle + c_2 |\psi_2\rangle + c_3 |\psi_3\rangle$. (If the eigenbasis is not orthonormal, we can always make it so via the Graham Schmidt procedure).

$$= \hat{\Omega}(c_1 |\psi_1\rangle + c_2 |\psi_2\rangle + c_3 |\psi_3\rangle)$$

We can distribute the operator into the sum:

$$= c_1\hat{\Omega} |\psi_1\rangle + c_2\hat{\Omega} |\psi_2\rangle + c_3\hat{\Omega} |\psi_3\rangle$$

The effect of the operator on an eigenvector is to produce the corresponding eigenvalue

$$= c_1\omega_1 |\psi_1\rangle + c_2\omega_2 |\psi_2\rangle + c_3\omega_3 |\psi_3\rangle$$

Meanwhile the c_i are just the inner product (overlap) with the state $c_i = \langle\omega_i|\psi\rangle$. Since these are just numbers, we can move them to the right. After breaking the inner product into separate bras and kets we have

$$= \omega_1 |\omega_1\rangle\langle\omega_1| |\psi\rangle + \omega_2 |\omega_2\rangle\langle\omega_2| |\psi\rangle + \omega_3 |\omega_3\rangle\langle\omega_3| |\psi\rangle$$

Finally we can remove the state upon which the operator acts from both sides to arrive at the final expression

$$\hat{\Omega} = \omega_1 |\omega_1\rangle\langle\omega_1| + \omega_2 |\omega_2\rangle\langle\omega_2| + \omega_3 |\omega_3\rangle\langle\omega_3| \quad \blacksquare$$

QUESTION 4 – 15 MARKS

Consider a spin-1/2 particle with a magnetic moment $\mu = gq/(2m)$.

Questions: a) At time $t = 0$, the observable S_x is measured with the result $+\hbar/2$. What is the state vector $|\psi\rangle$ immediately after the measurement? Explain your answer in a couple of sentences.

First of all, I am confused by this notation. Normally we would say that the *gyromagnetic ratio* is

$$\gamma = \frac{gq}{2m}$$

and that the magnetic moment is

$$\mu = \gamma S.$$

I will assume that this is what the problem meant.

When the measurement is made the state collapses onto the ψ_{x+} state. In the normalized z-representation this can be written

$$\boxed{|\psi(0)\rangle = \psi_{x+} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}}$$

b) Immediately after that S_x measurement, a magnetic field $\boldsymbol{B} = B_0\hat{z}$ is applied, with the quantum state of the particle henceforth evolving in time until a time T. What is the state of the system at time $t = T$?

We start by finding the Hamiltonian:

$$H = -\mu \cdot \mathbf{B} = \gamma \mathbf{B} \cdot \mathbf{S} = \frac{gq}{2m} B_0 S_z = \frac{gq}{2m} B_0 \frac{\hbar}{2}\sigma_z = -\frac{gq\hbar B_0}{4m}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

The time evolution operator is given by

$$e^{-iHt/\hbar} = \begin{bmatrix}e^{i\frac{gqB_0}{4m}t} & 0\\0 & e^{-i\frac{gqB_0}{4m}t}\end{bmatrix}$$

We can simplify this by rewriting in terms of the Larmor frequency

$$\omega_0 \equiv \gamma B_0 = \frac{gqB_0}{2m}$$

to get

$$e^{-iHt/\hbar} = \begin{bmatrix} e^{\frac{i\omega_0 t}{2}} & 0 \\ 0 & e^{\frac{-i\omega_0 t}{2}} \end{bmatrix}$$

When this operates on the initial state $|\psi(0)\rangle$ we get

$$\begin{aligned} |\psi(T)\rangle &= e^{-iHT/\hbar} |\psi(0)\rangle \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{\frac{i\omega_0 T}{2}} & 0 \\ 0 & e^{-\frac{i\omega_0 T}{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{\frac{i\omega_0 T}{2}} \\ e^{-\frac{i\omega_0 T}{2}} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \left[e^{\frac{i\omega_0 T}{2}} |\uparrow\rangle + e^{-\frac{i\omega_0 T}{2}} |\downarrow\rangle \right] \end{aligned}$$

at some specific time $t = T$.

c) At time $t = T$, the magnetic field is very quickly changed to $\boldsymbol{B} = B_0 \hat{y}$. After another time interval T, a measurement of S_x is carried out once more. What is the probability that a value $+\hbar/2$ is found?

The new Hamiltonian is

$$H^\star = -\hat{\mu} \cdot \mathbf{B} = -\frac{gq\hbar B_0}{4m} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -\frac{\omega_0 \hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

So we must calculate

$$|\psi(2T)\rangle = e^{-iH^\star T/\hbar} |\psi(T)\rangle = e^{\frac{i\omega_0 T}{2}} |\psi(T)\rangle$$

In order to exponentiate the Hamiltonian we rewrite the Pauli matrix using the diagonalization procedure:

$$e^{\sigma} = P e^{D} P^{-1}$$

where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is the diagonal matrix whose entries are the eigenvalues of σ_y and

$$P = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$$

are the matrices whose columns comprise the corresponding eigenvectors and its inverse, respectively. Therefore

$$\begin{aligned} |\psi(2T)\rangle &= e^{-iH^{\star}T}\,|\psi(T)\rangle \\ &= \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{\frac{i\omega_0 T}{2}} & 0 \\ 0 & e^{-\frac{i\omega_0 T}{2}} \end{bmatrix} \frac{1}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} e^{\frac{i\omega_0 T}{2}} \\ e^{-\frac{i\omega_0 T}{2}} \end{bmatrix} \end{aligned}$$

which after some trigonometry and a fair amount of simplification (re-normalization is not necessary since $e^{iH^{\star}T}$ is unitary) comes to

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 + i\sin(\omega_0 T) \\ \cos(\omega T) \end{bmatrix}$$

We want the probability that the particle is in the $s_x = \frac{\hbar}{2}$ state, so we need to calculate

$$\begin{aligned} \langle P_{x+}\rangle &= \langle \psi(2T)|P_{x+}|\psi(2T)\rangle \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 - i\sin(\omega_0 T) & \cos(\omega_0 T) \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 + i\sin(\omega_0 T) \\ \cos(\omega T) \end{bmatrix} \end{aligned}$$

where

$$P_{x+} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is the projection matrix for the ψ_{x+} state expressed in the z-basis. This gives a final result of

$$\langle P_{x+}\rangle = \frac{1}{2} + \frac{1}{2}\cos^2(\omega_0 T) = \boxed{\frac{3}{4} + \frac{1}{4}\cos(2\omega_0 T)}$$

So the probability oscillates between 0.5 and 1 with a frequency equal to twice the Larmor frequency.

QUESTION 5 – 15 MARKS

Consider a nickel Ni^{2+} ion with spin $S = 1$ embedded in an insulating crystalline host that produces a so-called crystal-field Hamiltonian

$$\mathcal{H}_{\mathrm{cf}} = \Delta S_z^2,$$

acting on the spin of the nickel ion. The crystalline field energy scale $\Delta = +10$ meV. The system is at time $t = 0$ in its minimum energy state (i.e. *ground state*) of $\mathcal{H}_{\mathrm{cf}}$.

At time $t = 0$ a magnetic field $\boldsymbol{B} = B_0\hat{x}$ applied in the x direction is turned on. The Zeeman Hamiltonian describing the interaction of the nickel ion with the magnetic field is

$$\mathcal{H}_{\mathrm{Zeeman}} = 2\mu_{\mathrm{B}}\boldsymbol{S}\cdot\boldsymbol{B},$$

where μ_{B} is the Bohr magneton.

Question: What is the probability that the system is still in its ground state 30 ps after a magnetic field $B_0 = 1$ T has been turned on?

$$|\psi_i\rangle \xrightarrow{t=0} V(t) \xrightarrow{\Delta T} |\psi_f\rangle \; ?$$

Transition amplitude: $a_{if} = \langle \psi_i(t=0) | \psi_f(t=\Delta T)\rangle$

$$a_{if} = \frac{1}{i\hbar}\int_0^{\Delta T} d\tau \, \langle \psi_f | V(\tau) | \psi_i \rangle \, e^{i\omega_{if}\tau} \quad , \quad \omega_{if} = \frac{E_f - E_i}{\hbar}$$

Transition probability:

$$P_{if} = |a_{if}|^2 = \frac{1}{\hbar^2}\left| \int_0^{\Delta T} d\tau \, \langle \psi_f | V(\tau) | \psi_i \rangle \, e^{i\omega\tau} \right|^2$$