Q: Find the probability density function for a particle undergoing simple harmonic motion.

A: We are looking for a function p(x) that satisfies

$$P(a < x < b) = \int_a^b p(x)\,\mathrm{d}x \tag{1}$$

where $P(a < x < b)$ is the probability that when the particle is observed at random that it will be found between positions a and b. The probability that a particle is will be observed in a certain range of positions is equal to the fraction of the time spent at that position. Therefore, for a small time interval $\mathrm{d}t$ the corresponding probability $\mathrm{d}p$ is

$$\mathrm{d}p = \frac{\mathrm{d}t}{T} \tag{2}$$

where T is the time required to move from $x = -A$ to $x = A$, i.e. half of the period. We can change variables to rewrite this in terms of position by using the velocity relation $\mathrm{d}x = v\,\mathrm{d}t$.

$$P(a < x < b) = \int_{x=a}^{x=b} \mathrm{d}p = \int_a^b \frac{\mathrm{d}x}{Tv} \tag{3}$$

For an object in SHM,

$$x(t) = A\sin(\omega t) \quad \text{and} \quad v(t) = A\omega\cos(\omega t) \tag{4}$$

while $T = \frac{\pi}{\omega}$. Substituting these into (3) gives

$$P(a < x < b) = \int_a^b \frac{\mathrm{d}x}{\pi A\cos(\omega t)} \tag{5}$$

Comparing this with equation (1) above, we see that

$$p(x) = \frac{1}{\pi A\cos(\omega t)} \tag{6}$$

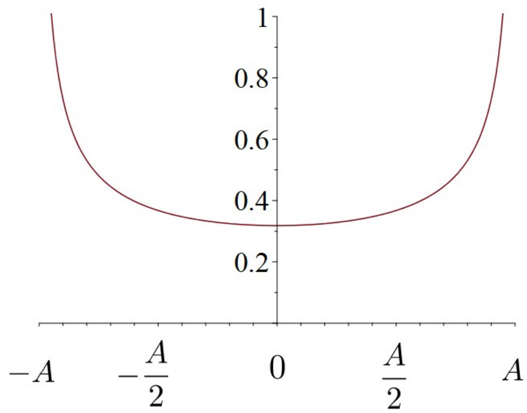
All that remains is to rewrite the trig expression in terms of x using that fact that

$$\cos(\omega t) = \sqrt{1 - \sin^2(\omega t)} = \sqrt{1 - \left(\frac{x}{A}\right)^2} \tag{7}$$

The final result is:

$$\boxed{p(x) = \frac{1}{\pi\sqrt{A^2 - x^2}}} \tag{8}$$

We can verify that integrating this from $x = -A$ to $x = A$ equals 1 as expected for a probability density function. Here is its graph showing that the particle is least likely to be observed near the equilibrium point—since that is where the particle is moving the fastest.



Q: What is the expectation value of the distance of the particle from the equilibrium position?

A: The expectation value of an arbitrary function $f(x)$ can be written

$$\langle f \rangle = \int_{-\infty}^{\infty} f(x) p(x)\,\mathrm{d}x \tag{9}$$

where $p(x)$ is the probability density function. In this case we have

$$\langle |x| \rangle = \int_{-A}^{A} |x| p(x)\,\mathrm{d}x = 2\int_{0}^{A} x p(x)\,\mathrm{d}x \tag{10}$$

where we have taken advantage of the fact that $p(x)$ is symmetrical with respect to $x = 0$. Substituting the earlier result (8), we have

$$\langle |x| \rangle = 2\int_{0}^{A} \frac{x}{\pi\sqrt{A^2 - x^2}}\,\mathrm{d}x \tag{11}$$

This can be computed using a straightforward u-substitution to get

$$\boxed{\langle |x| \rangle = \frac{2A}{\pi}} \tag{12}$$

As expected, the answer is closer to A than to 0.

Q: What is standard deviation of the particle's position?

A: For this we can use the equation

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \tag{13}$$

The second term inside the radical is zero due to the symmetry of the probability distribution function (8), so we just need to compute

$$\sigma_x = \left(\int_{-A}^{A} x^2 p(x)\, \mathrm{d}x \right)^{1/2} = \left(2 \int_0^A \frac{x^2}{\pi \sqrt{A^2 - x^2}} \right)^{1/2} \tag{14}$$

The integral can be performed using the trigonometric substitution $x = A \sin u$. I will omit the details, but the final result is:

$$\boxed{\sigma_x = \frac{A}{\sqrt{2}}} \tag{15}$$