Spin - $\frac{3}{2}$

Consider a spin$-\frac{3}{2}$ particle. In the z-representation the S_z operator is simply the diagonal matrix whose columns are the eigenvectors and whose nonzero entries are the corresponding eigenvalues.

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

In this representation the $m_s = \frac{3}{2}$ state is given by

$$\left|\frac{3}{2}\right\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can see that if we measure the spin in the z-direction the state is unchanged—since $\left|\frac{3}{2}\right\rangle$ is an eigenvalue of S_z—and that the eigenvalue is the value of the spin itself, $\frac{3\hbar}{2}$.

$$S_z\left|\frac{3}{2}\right\rangle = \frac{\hbar}{2}\begin{bmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{\hbar}{2}\cdot\begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{3\hbar}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This is consistent with the standard measurement paradigm in quantum mechanics:

$$\text{op}\,|\text{state}\rangle = \text{measurement}\,|\text{state}\rangle$$

We can also calculate the spin expectation value of the system when prepared in the $m_s = \frac{3}{2}$ state.

$$\langle S_z \rangle = \left\langle \tfrac{3}{2} \right| S_z \left| \tfrac{3}{2} \right\rangle$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \frac{3\hbar}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \boxed{\frac{3\hbar}{2}}$$

Since the state is a pure state and the act of measurement does not change it, the expectation value is again just the corresponding eigenvalue.

The raising and lowering (i.e. ladder) operators for S_z must obey

$$S_\pm \left| s, m_s \right\rangle = \left| s, m_s \pm 1 \right\rangle$$

This means they will have one nonzero diagonal one space above or below the main diagonal, and the terms must be chosen such that the overall spin quantum number $s = \sqrt{m_s(m_s + 1)}$ remains constant. For the $s = \frac{3}{2}$ system, we have

$$S_+ = \hbar \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S_- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

When the lowering operator acts on the $m_s = \frac{3}{2}$ state, we get

$$S_- \left| \frac{3}{2} \right\rangle = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sqrt{3} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

We can see that this is *not* an eigenvalue equation since the new state is different from the original.

Now we use the ladder operators to derive the S_x and S_y operators which are given by

$$S_x = \frac{1}{2}(S_+ + S_-) \text{ and } S_y = \frac{1}{2i}(S_+ - S_-).$$

The result is

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \quad \text{and} \quad S_y = \frac{\hbar}{2i}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & 2 & 0 \\ 0 & -2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{bmatrix}$$

To verify that this works, let's calculate the expectation value of the x-component of spin for the $m_s = \frac{3}{2}$ state:

$$\langle S_x \rangle = \left\langle \frac{3}{2} \middle| S_x \middle| \frac{3}{2} \right\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

As one would expect, when the system is prepared in a pure spin-z state, we have complete lack of information about the value of the spin in an orthogonal direction.

Next let's find the eigenvalues and eigenvectors for S_x. Supposing an arbitrary eigenstate

$$|\chi\rangle = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{bmatrix}$$

we can write

$$S_x |\chi\rangle = s_x |\chi\rangle$$

or

$$\frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}\begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{bmatrix} = s_x \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{bmatrix}$$

Defining λ such that $s_x = \frac{\hbar}{2}\lambda$ and inserting the 4x4 identity matrix, this becomes

$$\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{bmatrix}$$

which is equivalent to

$$\begin{bmatrix} -\lambda & \sqrt{3} & 0 & 0 \\ \sqrt{3} & -\lambda & 2 & 0 \\ 0 & 2 & -\lambda & \sqrt{3} \\ 0 & 0 & \sqrt{3} & -\lambda \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The only way this can be true is if $|\chi\rangle = 0$ or if the matrix is degenerate. Assuming the latter, we caclulate

$$\begin{vmatrix} -\lambda & \sqrt{3} & 0 & 0 \\ \sqrt{3} & -\lambda & 2 & 0 \\ 0 & 2 & -\lambda & \sqrt{3} \\ 0 & 0 & \sqrt{3} & -\lambda \end{vmatrix} = \lambda^4 - 10\lambda^2 + 9 = 0$$

so that

$$\lambda = \pm 1 \text{ or } \pm 3 \text{ and } s_x = \pm\frac{\hbar}{2} \text{ or } \pm\frac{3\hbar}{2}.$$

The corresponding (normalized) eigenvectors are

$$\left|\frac{3}{2}\right\rangle_x = \frac{1}{\sqrt{8}}\begin{pmatrix} 1 \\ \sqrt{3} \\ \sqrt{3} \\ 1 \end{pmatrix}, \quad \left|\frac{1}{2}\right\rangle_x = \frac{1}{\sqrt{8}}\begin{pmatrix} \sqrt{3} \\ 1 \\ -1 \\ -\sqrt{3} \end{pmatrix}, \quad \left|-\frac{1}{2}\right\rangle_x = \frac{1}{\sqrt{8}}\begin{pmatrix} \sqrt{3} \\ -1 \\ -1 \\ \sqrt{3} \end{pmatrix}, \quad \left|-\frac{3}{2}\right\rangle_x = \frac{1}{\sqrt{8}}\begin{pmatrix} 1 \\ -\sqrt{3} \\ \sqrt{3} \\ -1 \end{pmatrix}$$

Finally, let's calculate a projection operator. The projection operator

$$P_{-\frac{3}{2},x} = \left|-\frac{3}{2}\right\rangle_x {}_x\!\left\langle-\frac{3}{2}\right|$$

computes the probability that if the spin in the x-direction is measured the result will be $m_s = -\frac{3}{2}$. We calculate it as follows:

$$P_{-\frac{3}{2},x} = \left|-\frac{3}{2}\right\rangle_{x\;x}\left\langle-\frac{3}{2}\right| = \frac{1}{\sqrt{8}}\begin{bmatrix}1 & -\sqrt{3} & \sqrt{3} & -1\end{bmatrix}\frac{1}{\sqrt{8}}\begin{bmatrix}1\\ -\sqrt{3}\\ \sqrt{3}\\ 1\end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix}1 & -\sqrt{3} & \sqrt{3} & -1\\ -\sqrt{3} & 3 & -3 & \sqrt{3}\\ \sqrt{3} & -3 & 3 & -\sqrt{3}\\ -1 & \sqrt{3} & -\sqrt{3} & 1\end{bmatrix}$$

This means that if a system is prepared (for instance) in the state

$$|\chi\rangle = \frac{1}{3}\begin{bmatrix}1\\2\\0\\2\end{bmatrix}$$

then the probability that the observed spin in the x-direction will be $m_s = -\frac{3}{2}$ is

$$\begin{aligned}
P\left(m_s = -\frac{3}{2}\right) &= \langle\chi|P_{-\frac{3}{2},x}|\chi\rangle \\
&= \frac{1}{3}\begin{bmatrix}1 & 2 & 0 & 2\end{bmatrix}\frac{1}{8}\begin{bmatrix}1 & -\sqrt{3} & \sqrt{3} & -1\\ -\sqrt{3} & 3 & -3 & \sqrt{3}\\ \sqrt{3} & -3 & 3 & -\sqrt{3}\\ -1 & \sqrt{3} & -\sqrt{3} & 1\end{bmatrix}\frac{1}{3}\begin{bmatrix}1\\2\\0\\2\end{bmatrix} \\
&= \frac{13+4\sqrt{3}}{48} \approx .415
\end{aligned}$$